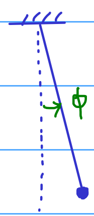
Lecture 14 (10/25/2019)

When we learn a numerical method, for example to find root of $f(x)=0$, we are naturally concerned about the effectiveness of the method. The following questions need to be addressed:

1) Does the sequence x_n converge?
   The bisection guarantees convergence, but Newton's method doesn't.
2) What is the order of convergence of x_n to a true root α? More generally, how fast does x_n converge to α?
   Newton method has order of convergence 2.
   Bisection method has order of convergence 1.
3) How long does it take to for a program to stop? Are the computer's memory and computing power sufficient to perform the task?

In real-life problems, the unknowns are usually functions, not just a real number. For example, in the pendulum problem, the unknown is the angle ϕ as a function of time. This unknown satisfies a differential equation



essentially of the form:

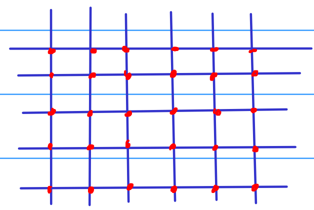
$$\underbrace{\phi'' + \phi}_{F(\phi)} = 0$$

Such a root_finding problem generally requires more sophisticated methods than the methods we will learn in this course. However, the idea we learn here is foundational to more complex methods in the future.

A famous numerical problem that requires more computing power than what is available is the Navier–Stokes equations:

$$\frac{\partial u}{\partial t} - \Delta u + u \nabla u + \nabla p = 0 .$$

The unknowns are three vector components $u = (u_1, u_2, u_3)$, each is a function of $x \in \mathbb{R}^3$ and time t, and the pressure function $p(x,t)$.

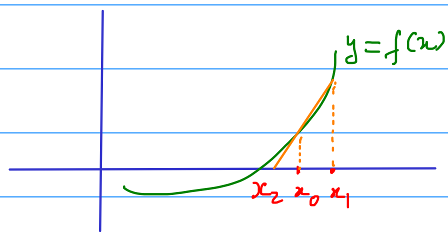


Since the computer can't store the information on functions u and p at infinitely many points (x,t), one has to "discretize" time and space by grid-points. Then one can find a numerical method to compute the values of u and p at the grid-points. In practice, the number of grid-points is too big for current computers to handle. (The problem of finding drag force of a flow of Reynolds number $\sim 10^5$ around a tennis ball would require $\sim 10^{15}$ grid points.)

4) Is the floating-point arithmetic error dominated by the error coming from the mathematical method?
We are often able to estimate the latter, but not the former.

* Secant method (also called 'chord method'):

The idea is as follows. Start with two points x_0 and x_1.

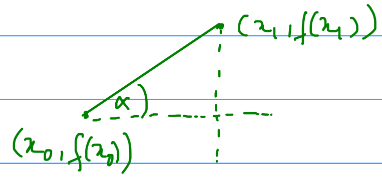


Draw a line (secant/chord) from two points $(x_0, f(x_0))$, $(x_1, f(x_1))$ on the graph of f. This line meets the x-axis at a point called x_2.

Then consider x_1 and x_2 as the starting points. Continue this procedure.

Let us make this idea rigorous. The line connecting $(x_0, f(x_0))$ and $(x_1, f(x_1))$ has slope

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



$m = \tan\alpha$

The equation of the line is

$$y - y_1 = m(x - x_1).$$

To find the intersection with the x-axis, we set $y = 0$ and solve for x:

$$x = x_1 - \frac{y_1}{m} = x_1 - y_1 \frac{x_1 - x_0}{y_1 - y_0}$$

Thus,

$$x_2 = x_1 - f(x_1) \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

In general,

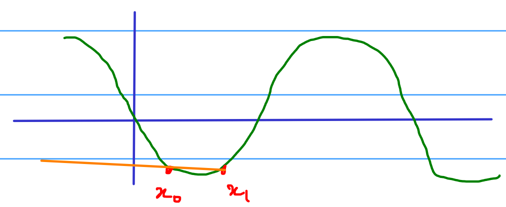
$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

The secant method has order of convergence $\frac{1+\sqrt{5}}{2}$.

It is an interesting problem how to get such a strange result. We will not derive it in the lecture, but refer to the textbook instead.

$$\underbrace{1}_{\text{order of bisection}} < \underbrace{\frac{1+\sqrt{5}}{2}}_{\text{order of secant}} < \underbrace{2}_{\text{order of Newton}}$$

Like Newton's method, the secant method doesn't guarantee convergence.



* Fixed point method:

The problem of finding root x of function f can be formulated as finding a fixed point of some function g.

$$g(x) = x.$$

Continue next time. See worksheet for a practice problem.