Lecture 15 (10/28/2019)

* Note on the order of convergence of Newton's method:
  - If $f'(\alpha) \neq 0$ (where α is the true root) then the order of convergence is $p=2$. In Lecture 12 on 10/21/2019, we derived

$$x_{n+1} - \alpha \approx -\frac{f''(\alpha)}{2f'(\alpha)}(x_n - \alpha)^2.$$

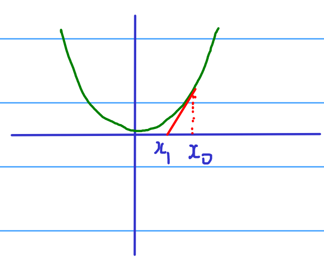
the condition $f'(\alpha) \neq 0$ is needed for that derivation to work.

  - If $f'(\alpha) = 0$, the convergence may be of a lower order.

Ex :

$$f(x) = x^2$$

f has only one root ($x=0$). At this root, $f'(0) = 0$.



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2}{2x_n} = \frac{x_n}{2}$$

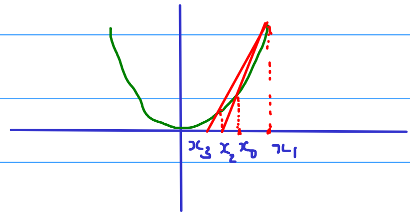
Thus,

$$\underset{\text{true root}}{x_{n+1} - 0} = \underbrace{\frac{1}{2}}_{\text{linear rate of convergence}}(x_n - 0)^{\overset{\text{order of convergence}}{1}}$$

We see that the vanishing of f' at 0 causes the order of convergence to drop from 2 to 1.

* Secant method:

The geometric description of secant method is similar to Newton's method.



The iterative formula is

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

which can be rewritten as

$$x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

In Newton's method, the numerator is replaced by $f'(x_n)$. At each, one needs to compute $f(x_n)$ and $f'(x_n)$. For secant method, one only needs to compute $f(x_n)$ ($f(x_{n-1})$ is considered as given from the previous step). Therefore, it takes less time for secant method to do one loop. However, it usually takes more loops in secant method than in Newton method for the error to be less than some prescribed error because secant method has lower order of convergence.

* Fixed point method:

A rootfinding problem can be formulated as a fixed point problem: find x such that $x = g(x)$.

Ex:

$$\underbrace{x^2 - 5}_{f(x)} = 0 \iff x = \underbrace{5/x}_{g(x)}$$

$$\underbrace{x^2 - 3x + 2}_{f(x)} = 0 \iff x = \underbrace{x^2 - 2x + 2}_{g(x)}$$

$$\underbrace{x^5 - 3x + 1}_{f(x)} = 0 \iff x = \underbrace{\frac{1}{3}(x^5 + 1)}_{g(x)}$$

To find a fixed point of $g(x)$, one can use the following iterative procedure:

- Take some x_0 close to the fixed point.
- Compute consecutively $x_1 = g(x_0)$, $x_2 = g(x_1)$, $x_3 = g(x_2)$, ...
  In general, $x_{n+1} = g(x_n)$.

Suppose the sequence x_n converges to some α. Then

$$\alpha = \lim x_{n+1} = \lim g(x_n) = g(\alpha)$$

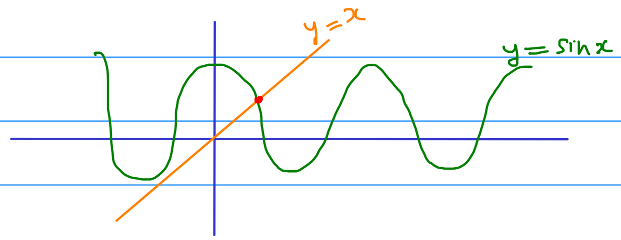
provided that g is continuous.

Thus, the limit α is a fixed point of g.

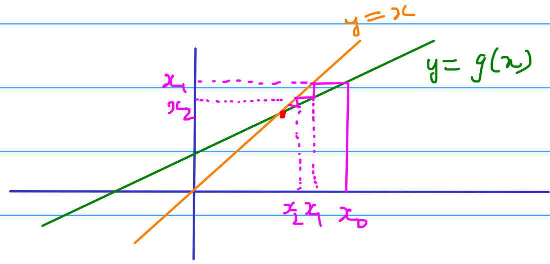
To illustrate this method, let's press $\boxed{1}$ (radian unit) on a calculator. Then press $\boxed{\cos}$ many times. After a few times, the calculator seems to give a convergent sequence. In this process, $g(x) = \cos x$.

$$x_0 = 1,$$
$$x_1 = g(x_0) = \cos 1 \approx 0.54030$$
$$x_2 = g(x_1) = \cos 0.54030 \approx 0.85755$$
$$x_3 = g(x_2) = \cos 0.85755 \approx 0.654290$$
$$x_4 = g(x_3) \approx 0.793480$$
$$x_5 = g(x_4) \approx 0.701368 8$$
$$x_6 = g(x_5) \approx 0.763960$$
$$x_7 = g(x_6) \approx 0.722102$$
$$x_8 = g(x_7) \approx 0.750417$$
$$x_9 = g(x_8) \approx 0.731404$$

The limit of this sequence is a fixed point of $g(x) = \cos x$, which is about 0.739085 13.



The iteration $x_{n+1} = g(x_n)$ can be illustrated visually by a "cobweb" diagram as follows:



- Draw the line (L): $y=x$ and the curve (C): $y=g(x)$.
- Start at x_0 on the x-axis. Draw a vertical line until intersecting (C).
- From the intersection point, draw a horizontal line until intersecting (L).
- From the intersection point, draw a vertical line until intersecting (C).
- ... (continue the procedure)

In many cases, the intersection points converge to the fixed point (the red dot).