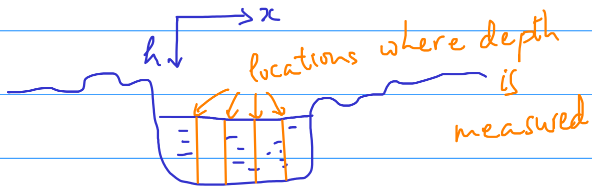
Lecture 16 (11/04/2019)

We have learned the following topics:

- Approximate the values of an arbitrary function using Taylor polynomials.
- How computers deal with numbers and basic operations. They only represent numbers approximately in floating-point format.
- How to find approximate roots and fixed points of a function.

Our next topic is rather qualitative than quantitative. It is the interpolation problem. One encounters an interpolation problem when the available data is insufficient.

Ex:



To measure the depth of a lake as function $h = h(x)$, we measure the depth at finitely many locations, for example at $x_1, x_2, \dots, x_n$. Then we have the data $h(x_1), h(x_2), \dots, h(x_n)$. To construct the function h from these data, we are dealing with an interpolation problem (curve fitting problem).

Ex: Suppose we have a 4x4 digital image. Each pixel contains a data, which is the gray scale at that position.

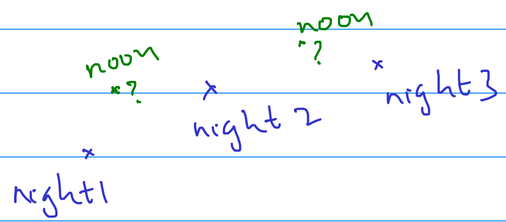
10	15	35	34
30	40	50	60
70	80	100	37
87	23	15	26

We want to zoom in, i.e. to magnify this image, by 150%.
The new image is of dimension 6×6 (36 pixels).

10	?	15			
		80	?	100	

There are $36-16=20$ new pixels. How to find the gray scale levels for these pixels? This is an interpolation problem.

Ex: The interpolation problem is not new. It arised as soon as people wanted to predict the positions of celestial objects. For example, the position of a star is observed and marked each night.

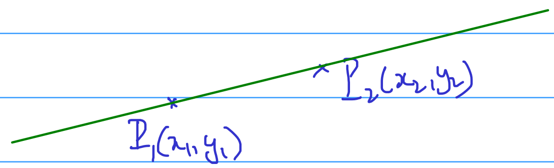


What are the position of the star at noon between night 1 and night 2?

There are many ways to interpolate a set of data. Let us consider one of the simplest ways. This is based on the idea of E. Waring (1779), L. Euler (1783), and J. Lagrange (1795).

To illustrate the idea, let's consider a few motivating examples. Given two points P_1 and P_2 on the plane. Find a function f such that

$$\begin{cases} f(x_1) = y_1, \\ f(x_2) = y_2. \end{cases}$$

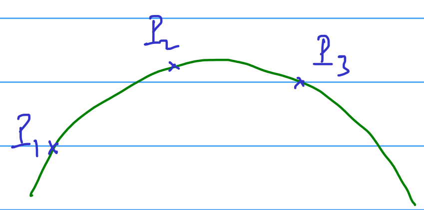


In other words, the graph of f must pass through P_1 and P_2. The most natural chose of f is perhaps a polynomial $f(x) = ax + b$, whose graph is a straight line. Here a and b must satisfy

$$\begin{cases} ax_1 + b = y_1 \\ ax_2 + b = y_2 \end{cases}$$

This is a system of two equations and two unknowns. Such a system typically has a unique solution (a, b).

How about 3 points P_1, P_2, P_3?



A line can't pass through 3 points (unless they are colinear). One can suggest that f is a quadratic polynomial.

$$f(x) = ax^2 + bx + c.$$

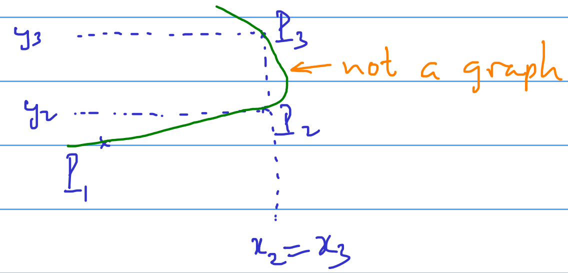
The coefficients a, b, c are to be determined. They satisfy

$$\begin{cases} ax_1^2 + bx_1 + c = y_1 \\ ax_2^2 + bx_2 + c = y_2 \\ ax_3^2 + bx_3 + c = y_3 \end{cases}$$

This is a linear system of three eqs. and three unknowns. It usually has a unique solution (a, b, c).

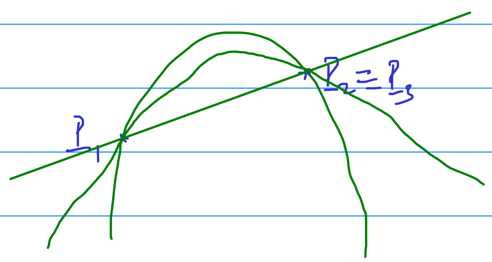
Is there any case when the problem has no solutions or infinitely many solutions?

- No solutions:



This happens when two points are vertically aligned with the each other: $\begin{cases} x_i = x_j \\ y_i \neq y_j \end{cases}$ for some $i \neq j$.

- Infinitely many solutions:



This happens when two points coincide with each other.

<u>Theorem</u>

Let $P_1(x_1, y_1), \dots, P_n(x_n, y_n)$ be n distinct points on the plane such that no two points are vertically aligned with each other. Then there is a unique polynomial P with degree $\leq n-1$ such that

$$P(x_i) = y_i \quad \forall i = 1, 2, \dots, n.$$

How to find this polynomial P? Should we solve a linear system of n eqs. and n unknowns to solve for the coefficients of this polynomial?

The answer is No. Lagrange discovered a much simpler method to find P.

First, let's find the polynomials $L_1, L_2, \dots, L_n$, each of which is of degree $n-1$ such that

$$\begin{cases} L_1(x_1) = 1 \\ L_1(x_2) = 0 \\ \dots \\ L_1(x_n) = 0 \end{cases} \qquad \begin{cases} L_2(x_1) = 0 \\ L_2(x_2) = 1 \\ L_2(x_3) = 0 \\ \dots \\ L_2(x_n) = 0 \end{cases} \qquad \dots$$

Polynomial L_i is equal to 1 at x_i, and 0 at any other x_j. Since $x_2, x_3, \dots, x_{n-1}$ is a root of L_1, we have

$$L_1(x) = c(x-x_2)(x-x_3)\dots(x-x_n)$$

Because L_1 is of degree $n-1$, c must be a constant.
Because $L_1(x_1) = 1$, we get

$$c = \frac{1}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}$$

We have found L_1. Finding L_2 is similar. Since $x_1, x_3, \dots, x_n$ are roots of L_2, we can write

$$L_2(x) = d(x-x_1)(x-x_3)\dots(x-x_n).$$

Since $L_2(x_2) = 1$, we get

$$d = \frac{1}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}$$

Then the polynomial P is given by

$$P(x) = y_1 L_1(x) + y_2 L_2(x) + \dots + y_n L_n(x) \qquad (*)$$

This is a polynomial of degree $\leq n-1$ and satisfies

$$P(x_1) = y_1 \cdot 1 + y_2 \cdot 0 + \dots + y_n \cdot 0 = y_1$$

$$\dots$$

$$P(x_n) = y_n$$

RHS $(*)$ is a linear combination of $L_1, L_2, \dots, L_n$. One can check that $L_1, L_2, \dots, L_n$ are linearly independent. For this reason, $L_1, \dots, L_n$ are called Lagrange basis polynomials. (They form as basis of the vector space P_{n-1}.)