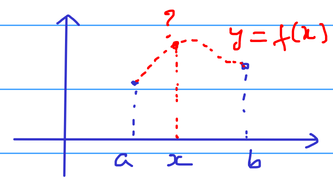
Lecture 17 (11/06/2019)

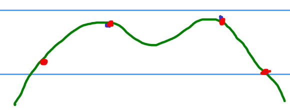
Interpolation is a type of problems that ask for the value of a function f at some point x in between two points a and b at which the values of f are known.



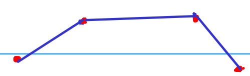
Interpolation problems are found in several different forms:

1) Data fitting:

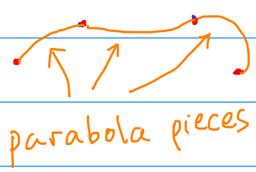
Given a set of points on the plane, find a curve (with certain properties) that passes through these points.



polynomial curve

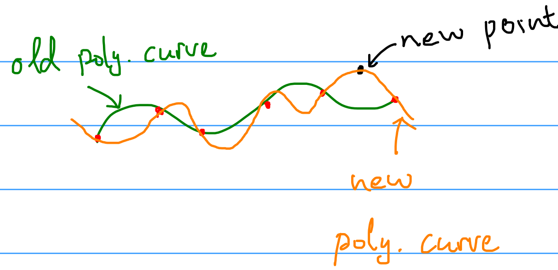


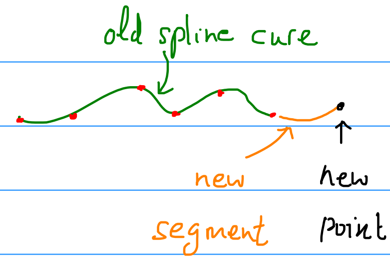
piecewise linear curve (known as linear spline curve)



quadratic spline curve

Linear spline is continuous, but not differentiable at the corner points. Quadratic polynomial is differentiable but not twice differentiable. In engineering, for example in designing car body or plane body, spline curves are more preferred than polynomial curves because having one more data point would require one to reconstruct the entire polynomial curve. On the other hand, one only needs to fix a local piece of the curve.

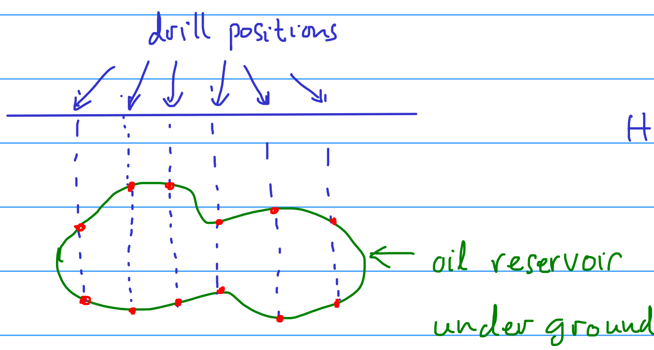




2) Reconstruction of function:

In many cases, we want to find not only a value, but a function, for example, the amount of rain as a function of time, the speed of a car as a function of time, ...

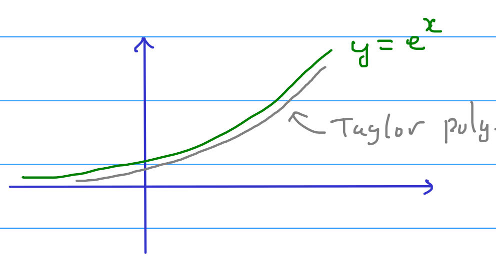
By sampling, we know the value of f at certain points, say $f(x_1), f(x_2), \dots, f(x_n)$. How to reconstruct f from these data?



How to reconstruct the shape of the oil reservoir from the red points?

3) Approximation:

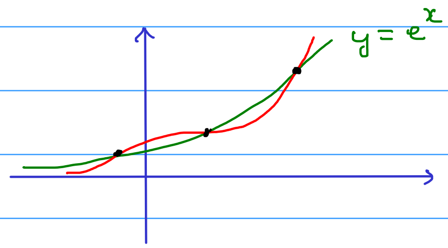
An example is the problem of finding a polynomial curve that approximates the graph of a given function.



We know that

$$e^x \approx \underbrace{1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}}_{\text{Taylor polynomial}}$$

Taylor polynomial approximates $f(x)$. But here we want a type about a different type of approximation: the polynomial curve must intersect the graph of f at a large amount of points.



In other words, find a poly. curve that passes through the points $(x_1, f(x_1)), (x_2, f(x_2)), \ldots, (x_n, f(x_n))$.

Polynomial interpolation

Given at set of distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ such that no two points are vertically align. Then there is a polynomial P of degree $\leq n-1$ such that

$$P(x_i) = y_i \qquad \forall\, 1 \leq i \leq n.$$

This polynomial is unique. Indeed, suppose Q is another polynomial that has the same property. Then

$$R = P - Q$$

has degree $\leq n-1$ and has n distinct roots $x_1, x_2, \ldots, x_n$. It must be the constant function 0. Thus, $P = Q$.

How to find P? There are many ways to do so. Two classical methods are by Lagrange and Newton respectively.

0) Direct method (not interesting):

Write $P(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then solve for $a_{n-1}, \ldots, a_1, a_0$ from n conditions $P(x_i) = y_i$. This amounts to solving a linear system of n equations and n unknowns. The drawback of this method is that it requires too much computation.

1) The method of Lagrange (1795):

$$P(x) = y_1 L_1(x) + y_2 L_2(x) + \cdots + y_n L_n(x)$$

where
$$L_1(x) = \frac{(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}$$

$L_2, L_3, \ldots, L_n$ are similarly defined.

The polynomials $L_1, L_2, \ldots, L_n$ form a basis for $P_{n-1}(\mathbb{R})$ (the vector space of all polynomials of degree $\leq n-1$).

In this methods, one needs to find the Lagrange basis polynomials $L_1, L_2, \ldots, L_n$ from the given data.

(See worksheet for an example).

2) The method of Newton (1687, in Principia Mathematica):

$$P(x) = c_0 \cdot 1 + c_1 N_1(x) + c_2 N_2(x) + \cdots + c_{n-1} N_{n-1}(x)$$

where

$$N_1(x) = x - x_1$$

$$N_2(x) = (x-x_1)(x-x_2)$$

$$\cdots$$

$$N_{n-1}(x) = (x-x_1)\cdots(x-x_{n-1})$$

The polynomials $1, N_1, N_2, \ldots, N_{n-1}$ form a basis for $P_{n-1}(\mathbb{R})$.

In this method, one needs to find the coefficients $c_0, c_1, \ldots, c_{n-1}$ from the given data.