Lecture 19 (11/13/2019)

Given n points $(x_1, y_1), \dots, (x_n, y_n)$, the Newton formula of the polynomial that fits these points is

$$P(x) = c_0 + c_1 N_1(x) + \dots + c_{n-1} N_{n-1}(x)$$

where
$$c_0 = P(x_1)$$
$$c_1 = P[x_1, x_2] := \frac{P(x_2) - P(x_1)}{x_2 - x_1}$$
$$c_2 = P[x_1, x_2, x_3] := \frac{P[x_2, x_3] - P[x_1, x_2]}{x_3 - x_1}$$

....

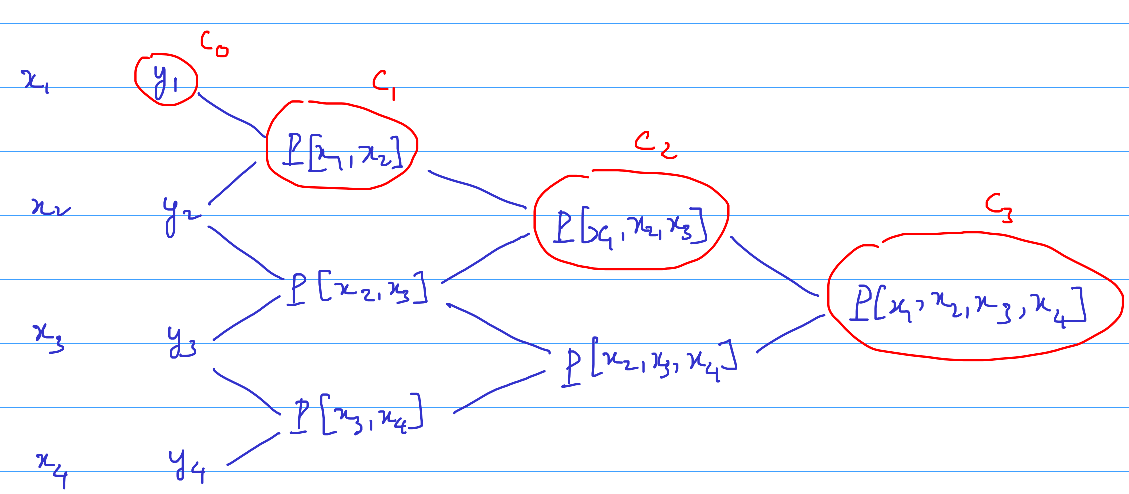
In general,
$$c_k = P[x_1, \dots, x_{k+1}] := \frac{P[x_2, \dots, x_{k+1}] - P[x_1, \dots, x_k]}{x_{k+1} - x_1}$$

This is called a divided difference.

The computation of $c_0, c_1, \dots, c_{n-1}$ can be illustrated in a diagram. Let's consider 4 points:

$$(x_1, y_1) = (1, 0)$$
$$(x_2, y_2) = (-1, 1)$$
$$(x_3, y_3) = (2, 3)$$
$$(x_4, y_4) = (3, 1)$$

Diagram:



Now plug in the given data into the diagram:

$$
\begin{array}{cccccccc}
1 & 0 = c_0 & & & & & & \\
 & & \dfrac{1-0}{-1-1} = -\tfrac{1}{2} = c_1 & & & & & \\
-1 & 1 & & \dfrac{2/3-(-1/2)}{2-1} = \tfrac{7}{6} = c_2 & & & & \\
 & & \dfrac{3-1}{2-(-1)} = \tfrac{2}{3} & & \dfrac{7/6-(-2/3)}{1-3} = -\dfrac{11}{12} = c_3 & & & \\
2 & 3 & & \dfrac{2/3-(-2)}{-1-3} = -\tfrac{2}{3} & & & & \\
 & & \dfrac{3-1}{2-3} = -2 & & & & & \\
3 & 1 & & & & & &
\end{array}
$$

Thus,

$$P(x) = c_0 + c_1(x-x_1) + c_2(x-x_1)(x-x_2) + c_3(x-x_1)(x-x_2)(x-x_3)$$

$$= 0 + \left(-\frac{1}{2}\right)(x-1) + \frac{7}{6}(x-1)(x+1) + \left(-\frac{11}{12}\right)(x-1)(x+1)(x-2)$$

$$= -\frac{11}{12}x^3 + 3x^2 + \frac{5x}{12} - \frac{5}{2}$$

Problem 4 and 5 in Homework 6 ask you to program Newton formula in Matlab. We start by programming the divided differences. This can be done because divided differences are defined recursively:

$$P[x_1, x_2, \dots, x_n] = \frac{P[x_2, \dots, x_n] - P[x_1, \dots, x_{n-1}]}{x_n - x_1}$$

* Question : how well does the interpolation polynomials approximate a given function f ?

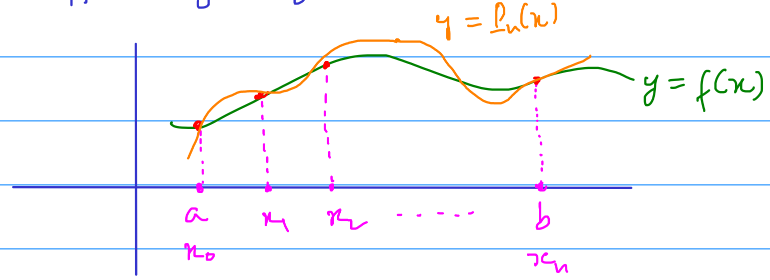
Let f be a function on an interval $[a,b]$.

We sample n points $x_1, x_2, \dots, x_n$ on the interval $[a,b]$, equally spaced.

These x_i's correspond to n points on the graph of f. Let P_n be the interpolation polynomial that fits these n points. The question is: is it true that the error

$$|f(x) - P_n(x)|$$

is less than some prescribed error ε, for all $x \in [a,b]$, for n sufficiently large?

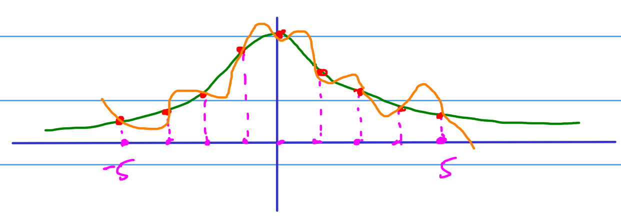


The answer depends on the function f and the interval $[a,b]$.

Let's consider an example.

$$f(x) = \frac{1}{1+10x^2}$$

This is of course not a polynomial. We sample n points $x_1, x_2, \dots, x_n$ on the interval $[-5,5]$, equally spaced.



Using Matlab with $n = 9$, we see that P_9 fluctuates a lot on the interval $[-5,5]$.

We will continue this experiment next time.