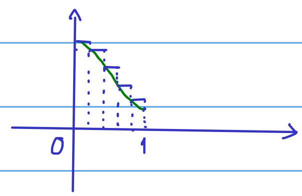
Lecture 2 (9/27/2019)

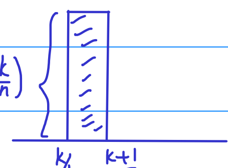
Last time, we posed the problem of evaluating the integral $\int_0^1 \cos(x^2)\,dx$ numerically.

This integral can be approximated by Riemann sum.



$f(x) = \cos(x^2)$

The area under the curve can be appr. by the sum of areas of the rectangles:



$$\int_0^1 f(x)\,dx \approx \sum_{k=0}^{n-1} \text{area of rectangle } k\text{'th}$$

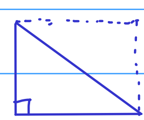
$$= \sum_{k=0}^{n-1} \frac{1}{n} f\left(\frac{k}{n}\right)$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \cos\left(\frac{k^2}{n^2}\right)$$

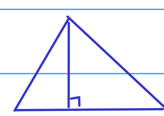
* Comment:

The idea of approximating the area of a general shape by the area of simpler shapes dates back to Archimedes (if not earlier). He used this idea to appr. the area of a circle.

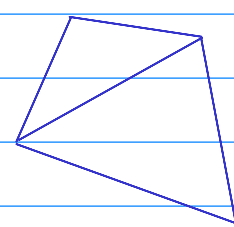
Note that only have in hand the area of a square. Then area of rectangle is obtained by concatenating many squares together.



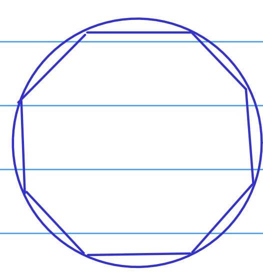
right triangle



triangle



quadrilateral



circle (approximation)

With $n = 1000$, $\quad \displaystyle\int_0^1 \cos(x^2)\,dx \approx \frac{1}{1000} \sum_{k=0}^{999} \cos\left(\frac{k^2}{10^6}\right)$. $\qquad (*)$

Before the age of computer, one can only rely on desk calculators for basic operations such as addition, subtraction, multiplication and division. Thus, a desk calculator can only evaluate polynomials and rational functions (quotient of two polynomials). The approximation $(*)$ would not be satisfying at that time because it involves transcendental function (cosine).

An approximation method we already knew is approximation by Taylor polynomials. Recall:

If f is a $(n+1)$'st differentiable on an interval containing x_0 then

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_n(x)} + R_n(x)$$

where $\qquad R_n(x) = \dfrac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$

c_x is some number between x_0 and x.

This is known as Lagrange theorem.

Taylor appr. helps us evaluate approximately quite general functions:

$$f(x) = \underbrace{P_n(x)}_{\text{computable}} + \underbrace{R_n(x)}_{\text{error, hopefully estimable}}$$

If one knows how to control the size of $f^{(n+1)}$ then the error term $R_n(x)$ is under control.

Ex: Compute $\sqrt{8}$ with precision 10^{-9}.

We know that $\sqrt{9}=3$. Let's consider function

$$f(x) = \sqrt{x}$$

Then

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-3/2}$$

$$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-5/2}$$

....

$$f^{(n)}(x) = \underbrace{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\,?\right)}_{n \text{ terms}} x^{-?}$$

$$= \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right) x^{-\frac{1}{2}-(n-1)}$$

Therefore,

$$f^{(n)}(9) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right) 9^{-\frac{1}{2}-(n-1)}$$

Now we can apply Taylor appr. to f about $x_0 = 9$, with $x = 8$

$$f(8) = \underbrace{f(9) + \sum_{k=1}^{n} \frac{f^{(k)}(9)}{k!}(8-9)^k}_{p_n(8)} + R_n(8)$$

Let us simplify $p_n(8)$:

$$p_n(8) = 3 + \sum_{k=1}^{n} \frac{f^{(k)}(9)}{k!}(-1)^k$$

This sum is programmable in Matlab.

Let us estimate the error by Lagrange's theorem:

$$R_n(8) = \frac{f^{(n+1)}(c)}{(n+1)!}(8-9)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(-1)^{n+1}$$

Here c is some number between 8 and 9.

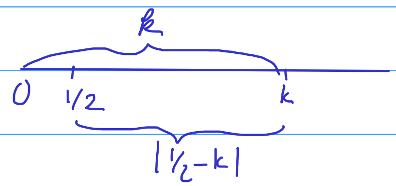
$$f^{(n+1)}(c) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n\right)c^{-\frac{1}{2}-n}$$

The size of $f^{(n+1)}(c)$ is bounded by

$$|f^{(n+1)}(c)| \le \frac{1}{2}\underbrace{\left|\frac{1}{2}-1\right|}_{\le 1}\cdot\underbrace{\left|\frac{1}{2}-2\right|}_{\le 2}\cdots\underbrace{\left|\frac{1}{2}-n\right|}_{\le n} 8^{-\frac{1}{2}-n}$$

$$= \frac{1}{2}n!\,8^{-\frac{1}{2}-n}$$

$$< \frac{1}{2}n!\,8^{-n}$$



Then $$|R_n(8)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} < \frac{\frac{1}{2}n!\,8^{-n}}{(n+1)!} = \frac{1}{2(n+1)}8^{-n}$$

This number goes to 0 quite rapidly as n goes to infinity.
If we have $\frac{1}{2(n+1)}8^{-n} < 10^{-9}$ then $|R_n(8)|$ is guaranteed to be under 10^{-9}. We can check with a calculator that $n=10$ will do it.