# Lecture 21 (11/18/2019)

* Remarks on Problem 4 of HW 6 :

Name the function that takes in a function $f$ and an array $x = (x_1, \dots, x_n)$ and gives the divided difference $f[x_1, \dots, x_n]$ by dd. Informly, one can write

$$dd[x_1, \dots, x_n] = \frac{dd[x_2, \dots, x_n] - dd[x_1, \dots, x_{n-1}]}{x_n - x_1}.$$

To be able to write Matlab code, one needs to know how to get the arrays $(x_2, \dots, x_n)$ and $(x_1, \dots, x_{n-1})$ from array $(x_1, \dots, x_n)$. There are many ways to do it. One of them is

```
n = length(x)
```

(*)
```
z = x
z(1) = []      // now z = (x2, x3, ..., xn)
w = x
w(n) = []      // now w = (x1, ..., xn-1)
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Then function dd can be written as

```
function y = dd(f, x)
    n = length(x)
    if n == 2
        y = (f(x(2)) - f(x(1))) / (x(2) - x(1))
    else
        ...the segment (*) goes in here....
        y = (dd(f, z) - dd(f, w)) / (x(n) - x(1))
end
```

Last time we obtained a formula of error of polynomial interpolation which looks similar to Lagrange's theorem for Taylor approximation:

Put $a = \min\{x_1, x_2, \dots x_n\}$,

$b = \max\{x_1, x_2, \dots, x_n\}$.

For each $x \in (a,b)$, there exists a number $r = r_x \in (a,b)$ such that

$$f(x) - P_n(x) = \frac{f^{(n)}(r_x)}{n!}(x-x_1)(x-x_2)\cdots(x-x_n)$$

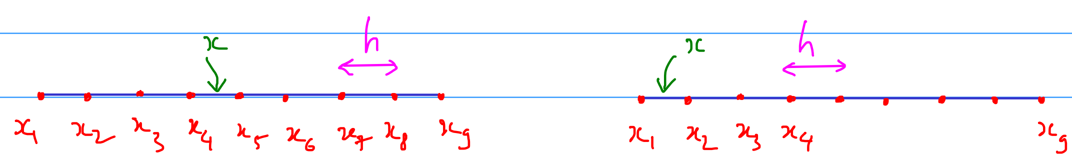
Take the absolute value of both sides:

$$|f(x) - P_n(x)| = \frac{|f^{(n)}(r_x)|}{n!}|x-x_1|\,|x-x_2|\cdots|x-x_n|$$

The size of the error $f(x) - P_n(x)$ depends on

- $n$,
- the size of $f^{(n)}$
- the size of $|x-x_1|\,|x-x_2|\cdots|x-x_n|$.

Let us investigate how big $|x-x_1|\cdots|x-x_n|$ can be. Suppose the sample points $x_1, x_2, \dots, x_n$ are evenly spaced, sorted ascending.




Consider two different positions of $x$ on the interval $[a,b]$.

- $x$ is near the middle of the interval
- $x$ is near the endpoints, say between $x_1$ and $x_2$.

(See the pictures, where $n = 9$.)

In the first case,

$$|x-x_4| \sim h$$
$$|x-x_5| \sim h$$
$$|x-x_3| \sim 2h$$
$$|x-x_6| \sim 2h$$
$$|x-x_2| \sim 3h$$
$$|x-x_7| \sim 3h$$

In the second case,

$$|x-x_1| \sim h$$
$$|x-x_2| \sim h$$
$$|x-x_3| \sim 2h$$
$$|x-x_4| \sim 3h$$
$$|x-x_5| \sim 4h$$
$$|x-x_6| \sim 5h$$

We see that the product $|x-x_1|\ldots|x-x_n|$ is larger on the second case than the first. This is because $x$ is "far" from the cluster. This is why the graph of function $(x-x_1)\ldots(x-x_n)$ oscilates irregularly largely near the endpoints of the interval.

To see how big $(x-x_1)\cdots(x-x_n)$ could be, we only need to look at the worse case (the second case). We see that

$$|x-x_1|\ldots|x-x_n| \leq h\,h\,(2h)\,(3h)\cdots((n-1)h)$$
$$= (n-1)!\, h^n$$

Therefore,

$$|f(x)-P_n(x)| \leq \frac{1}{n!}\left(\max_{[a,b]}|f^{(n)}|\right)(n-1)!\,h^n$$
$$= \frac{h^n}{n}\max_{[a,b]}|f^{(n)}|$$

Using the fact that $h = (b-a)/(n-1)$, we get

$$|f(x)-P_n(x)| \leq \frac{1}{n}\left(\frac{b-a}{n-1}\right)^n \max_{[a,b]}|f^{(n)}|$$

Ex: $f(x) = e^x$ and $[a,b] = [0,1]$.

Then $f^{(n)}(x) = e^x$ and $|f^{(n)}(x)| = e^x \le e^1 = e$.

Then

$$|f(x) - P_n(x)| \le \frac{1}{n}\left(\frac{1-0}{n-1}\right)^n e = \frac{e}{n}\frac{1}{(n-1)^n}.$$

To make sure that the error $|f(x) - P_n(x)|$ never exists $10^{-6}$, we only need to choose $n$ sufficiently large such that

$$\frac{e}{n}\frac{1}{(n-1)^n} < 10^{-6}.$$

Calculator yields $n \geq \dots$

Ex $f(x) = \sin x$ and $[a,b] = [0,\pi]$.

The key is to estimate $|f^{(n)}(x)|$. In this case,

$$f^{(n)}(x) \in \{\pm \cos x, \pm \sin x\}.$$

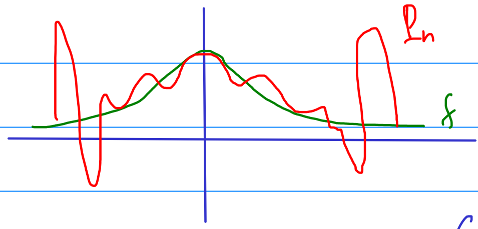
Then $|f^{(n)}(x)| \le 1$. Then

$$|f(x) - P_n(x)| \le \frac{1}{n}\left(\frac{\pi-0}{n-1}\right)^n \cdot 1 = \underbrace{\frac{1}{n}\left(\frac{\pi}{n-1}\right)^n}_{\text{independent of } x\text{, can be made arbitrarily small by increasing } n.}$$

Ex $f(x) = \dfrac{1}{1+x^2}$ and $[a,b] = [-5,5]$.

For this problem, one runs into the issue of estimating $|f^{(n)}(x)|$. This function in fact grows very quickly when $n$ increases. (See Matlab experiment.) In fact, one has

$$\max_{[-5,5]} |f(x) - P_n(x)| \longrightarrow \infty \quad \text{as } n \to \infty$$




The error is large near the end points. This is called Runge's phenomenon (discovered by Runge in 1901).