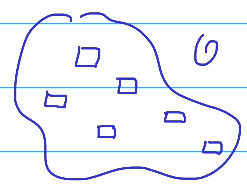
Lecture 22 (11/20/2019)

* Numerical integration: the problem of finding an approximate value of an integral $\int_a^b f(x)\,dx$ is prevalent in real life.

Measuring (length, weight, speed,...) is a form of taking integral. For example, an object $\mathcal{O}$ with mass density function

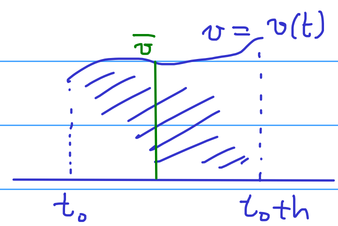


$\rho = \rho(x)$ has total mass

$$M = \int_{\mathcal{O}} \rho(x)\,dx.$$

To find M, we put $\mathcal{O}$ on a scale instead of evaluating the integral precisely.

Speedometer of a vehicle doesn't precisely show the real time speed, but rather an average speed over a very short amount of time.



Let v be the real time speed as a function of t. We don't know this function. However, speedometer can show us an average speed $\bar{v}$. From Math 252 (Integral Calculus), we learned that

$$\bar{v} = \frac{1}{h}\int_{t_0}^{t_0+h} v(t)\,dt.$$

Thus, speedometer gives us an (approximate) value of an integral.

* Even if we are given an explicit formula of the integrand, we may not be able to compute the exact value of the integral using the Fundamental Theorem of Calculus.

For example, the indefinite integrals

$$\int e^{-x^2}dx, \quad \int \frac{\sin x}{x}\,dx, \quad \int e^{e^x}dx, \ldots$$

cannot be written in finite form (instead of as a series) as a combination (i.e. addition, subtraction, multiplication, division, composition) of elementary functions. This is an observation of Joseph Louville (1830s). In 1960s, Robert Risch developed an algorithm to check if a function has an antiderivative that can be written as a combination in finite form of elementary functions.

Nevertheless, the integrals $\int_0^1 e^{-x^2}dx$, $\int_1^2 \frac{\sin x}{x}\,dx, \ldots$

are well-defined because the integrands are continuous. Since their antiderivative are not available to us, we need to find an approximation method to find these integrals.

* Even if it is possible to find the antiderivative of the integrand and use the Fundamental Theorem of Calculus. For example, to find $\int_0^1 x e^x dx$

one does integration by parts once. To find $\int_0^1 x^2 e^x dx$,

one does integration by parts twice. To find $\int_0^1 x^{20} e^x dx$,

one does integration by parts 20 times, etc. The process of finding antiderivative seems not efficient.

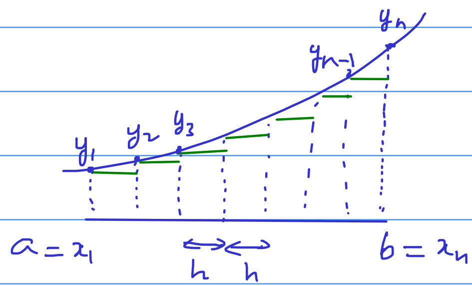
We will discuss several classical methods of numerical integration.

1) Riemann sum (and some variations of it)
2) Taylor polynomial approximation.
3) Interpolation polynomial (HW7, Prob. 2)
4) Gauss quadrature formula.

* Riemann sum:

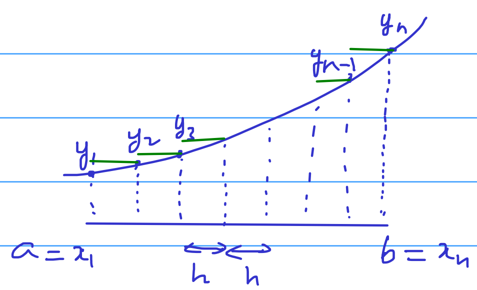
Let $f : [a,b] \to \mathbb{R}$ be a continuous function.

- Left-point rule:



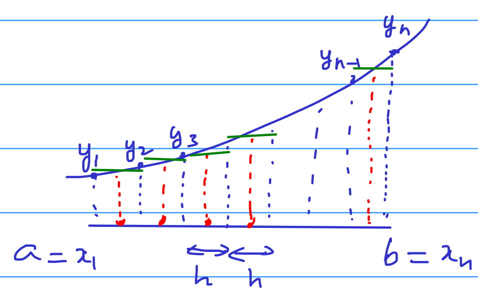
$$\int_a^b f(x)dx \approx h\, y_1 + h\, y_2 + \cdots + h\, y_{n-1}$$
$$= h(y_1 + y_2 + \cdots + y_{n-1}).$$

- Right-point rule:



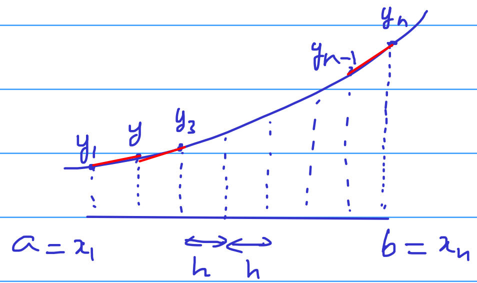
$$\int_a^b f(x)dx \approx h\, y_2 + h\, y_3 + \cdots + h\, y_n$$
$$= h(y_2 + y_3 + \cdots + y_n).$$

- Midpoint rule:



$$\int_a^b f(x)\,dx \approx h\, f\left(\frac{x_1+x_2}{2}\right) +$$
$$h\, f\left(\frac{x_2+y_3}{2}\right) +$$
$$\cdots + h f\left(\frac{x_{n-1}+x_n}{2}\right)$$

- Trapezoid rule :



$$\int_a^b f(x)\,dx \approx h\,\frac{y_1+y_2}{2} + h\,\frac{y_2+y_3}{2} + \cdots + h\,\frac{y_{n-1}+y_n}{2}$$

$$= \frac{h}{2}\left(y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n\right).$$

We will see that the trapezoid rule produces a better error than the three previous rules.