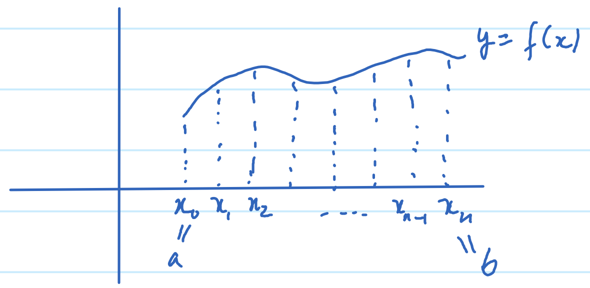
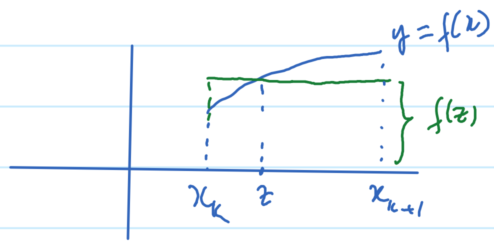
Lecture 23

November 22+25, 2019

Error estimates for left-point, right-point, midpoint method



Divide $[a,b]$ into n equal subintervals $[x_0,x_1], [x_1,x_2], \dots, [x_{n-1},x_n]$.
We approximate the integral over each subinterval $[x_k, x_{k+1}]$,
namely $\int_{x_k}^{x_{k+1}} f(x)\,dx$, by $h f(z)$ where z is a point in between x_k and x_{k+1}.



Area under the curve
$\approx$ area of the rectangle with width h and height $f(z)$.

$$z = \begin{cases} x_k & \text{for left-point rule,} \\ x_{k+1} & \text{for right-point rule,} \\ \dfrac{x_k + x_{k+1}}{2} & \text{for mid-point rule.} \end{cases}$$

We want to estimate the error $\varepsilon_k = \int_{x_k}^{x_{k+1}} f(x)\,dx - h f(z)$.

Note that one can write $h f(z)$ as $\int_{x_k}^{x_{k+1}} f(z)\,dx$.

Thus, the error ε_k can be written as

$$\varepsilon_k = \int_{x_k}^{x_{k+1}} (f(x) - f(z))\,dx.$$

We want to estimate ε_k in terms of h.

We use Taylor expansion of order 0 around z :

$$f(x) = \underbrace{p_0(x)}_{f(z)} + \underbrace{R_0(x)}_{f'(c_x)(x-z)}$$

Thus, $f(x) - f(z) = f'(c_x)(x-z)$.

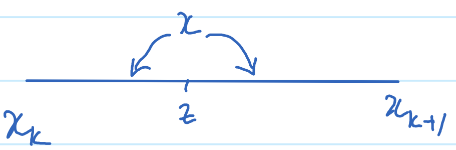
Denote $M = \max\limits_{x \in [a,b]} |f'(x)|$ We have

$$|f(x) - f(z)| = |f'(c_x)|\,|x-z| \leq M\,|x-z|.$$

We have

$$|\varepsilon_k| = \left| \int_{x_k}^{x_{k+1}} (f(x) - f(z))\,dx \right| \underbrace{\leq}_{\text{called triangle inequality}} \int_{x_k}^{x_{k+1}} |f(x) - f(z)|\,dx$$

$$\leq \int_{x_k}^{x_{k+1}} M\,|x-z|\,dx$$



We break this integral into two parts: integral over $[x_k, z]$ and integral over $[z, x_{k+1}]$.

$$\int_{x_k}^{x_{k+1}} |x-z|\,dx = \int_{x_k}^{z} (z-x)\,dx + \int_{z}^{x_{k+1}} (x-z)\,dx$$

$$= \frac{(z-x_k)^2}{2} + \frac{(x_{k+1}-z)^2}{2}. \qquad (*)$$

For left-point rule, $z = x_k$ and

$$(*) = \frac{0}{2} + \frac{(x_{k+1}-x_k)^2}{2} = \frac{h^2}{2}$$

For right-point rule, $z = x_{k+1}$ and

$$(*) = \frac{h^2}{2}$$

For midpoint rule, $z = (x_k + x_{k+1})/2$ and $(*) = \frac{h^2}{4}$.

Thus,

$$|\varepsilon_k| \le \begin{cases} \dfrac{Mh^2}{2} & \text{for left/right point rule.} \\ \dfrac{Mh^2}{4} & \text{for midpoint rule.} \end{cases}$$

This is error estimate on each subinterval $[x_k, x_{k+1}]$ The error on the whole interval $[a,b]$ is the accumulation of the errors on subintervals. Note that there are n subintervals. Thus,

$$\left| \int_a^b f(x)\,dx - \text{Riemann sum} \right| \le \begin{cases} Mn\dfrac{h^2}{2} & \text{for left/right} \ldots \\ Mn\dfrac{h^2}{4} & \text{for midpoint} \ldots \end{cases}$$

Note that $h = \dfrac{b-a}{n}$. Hence,

$$\left| \int_a^b f(x)\,dx - \text{Riemann sum} \right| \le \begin{cases} M\dfrac{(b-a)^2}{2}\dfrac{1}{n} & \text{for left/right} \ldots \\ M\dfrac{(b-a)^2}{4}\dfrac{1}{n} & \text{for midpoint} \ldots \end{cases}$$

We see that the error in the midpoint rule is slightly better than the error in the left/right point rule. With a more subtle analysis (using Taylor expansion of order 1 instead of 0), one can show that the error in midpoint method is at most

$$\tilde{M}\,\frac{(b-a)^3}{24}\,\frac{1}{n^2}$$

where $\tilde{M} = \max\limits_{[a,b]} |f''(x)|$.