Lecture 24

November 27, 2019

Last time we derived some error estimates for the left point, right point and midpoint rule. Accordingly, if we put

$$e_n = \left| \int_a^b f(x)\,dx - \text{Riemann sum} \right|$$

Then

$$e_n \le \underbrace{\frac{M(b-a)^2}{2}\frac{1}{n}}_{a_n} \qquad \text{where } M = \max_{[a,b]} |f'(x)|.$$

This suggests that as $n \to \infty$, e_n goes to 0 at a rate $1/n$ (or faster).
This is a polynomial rate, as oppose to exponential rate $(2^{-n}, e^{-n}, \dots)$

It is natural to ask what is the order of convergence of e_n to 0?
It is easier for us to find the order of convergence of a_n rather than e_n.

We want to find numbers p and C such that $a_{n+1} \le C a_n^p$.

Because

$$a_{n+1} \sim \frac{1}{n+1}$$

$$a_n^p \sim \frac{1}{n^p}$$

we see that for $a_{n+1} \le C a_n^p$ for n large, p is at most 1.
Therefore, a_n converges to 0 at order $p = 1$.

For the left/right point rule, $1/n$ is the optimal rate of decay of e_n.
In other word, one can find an example of a function f such that e_n goes to 0 no faster than $1/n$.

* For midpoint rule, however, $1/n$ is not an optimal rate of decay of e_n. By a more subtle error estimate (using Taylor expansion of degree 1), one can show that

$$e_n \le \underbrace{\frac{\tilde{M}(b-a)^3}{24} \frac{1}{n^2}}_{b_n} \qquad \text{where } \tilde{M} = \max_{[a,b]} |f''(x)|$$

This shows that e_n in fact goes to 0 at a rate of $1/n^2$ (or faster). However, the order of convergence of b_n is still equal to 1.

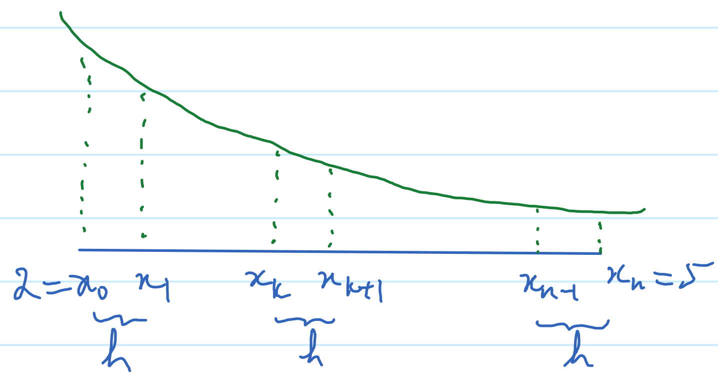
* For trapezoid rule, $\quad e_n \le \dfrac{\tilde{M}(b-a)^3}{12} \dfrac{1}{n^2}$

where $\tilde{M} = \max_{[a,b]} |f''(x)|$.

Ex: Compute approximately the integral

$$I = \int_2^5 \frac{1}{x}\,dx$$

using left point and trapezoid rule using $n+1$ equally spaced sample points $2 = x_0 < x_1 < . < x_n = 5$.



$$h = \frac{5-2}{n} = \frac{3}{n}$$

We have $x_0 = 2$, $x_1 = 2+h$, $x_2 = 2+2h$, ..., $x_k = 2+kh$, ..

Thus, $x_k = 2 + kh = 2 + \frac{3k}{n}$.

Left point:

$$\int_2^5 \underbrace{\frac{1}{x}}_{f(x)} dx \approx \sum_{k=0}^{n-1} \underbrace{h f(x_k)}_{\substack{\text{area of} \\ \text{rectangle over} \\ \text{the interval } [x_k, x_{k+1}]}} = \frac{3}{n} \sum_{k=0}^{n-1} \frac{1}{x_k}$$

$$= \frac{3}{n} \sum_{k=0}^{n-1} \frac{1}{2 + \frac{3k}{n}}.$$

Trapezoid:

$$\int_2^5 \frac{1}{x} dx \approx \sum_{k=0}^{n-1} h \left(\frac{f(x_k) + f(x_{k+1})}{2} \right)$$

$$= \frac{3}{2n} \sum_{k=0}^{n-1} \left(\frac{1}{x_k} + \frac{1}{x_{k+1}} \right).$$

* How to enter this sum in Matlab?

Note that Matlab indexes entries of an array by 1, 2, 3, ...
not 0, 1, 2, ...

Type h = 3/n

x = 2 : h : 5

At this time, Matlab understands that x(1) = 2, x(2) = 2+h, ...

We need to adjust $x_k = 2 + \frac{3(k-1)}{n}$ for $k = 1, 2, \ldots, n+1$.

See Matlab code on class website.