Lecture 26

Wednesday, December 4, 2019

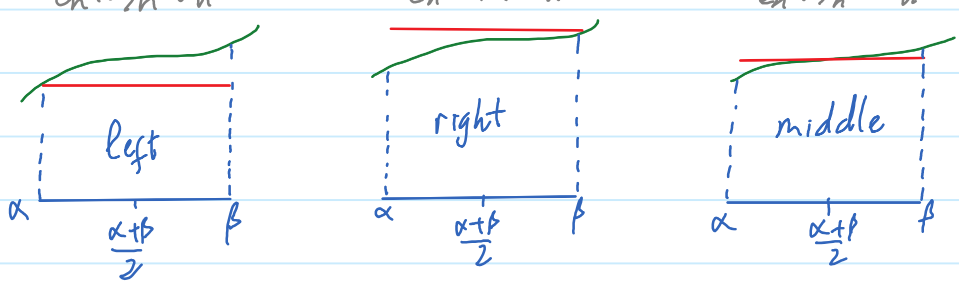
We discussed several methods to approximate definite integrals.

Put $$I = \int_a^b f(x)\,dx$$

This is the exact integral. We divide the interval $[a,b]$ into n equal subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$. On each of these subintervals, call $[\alpha, \beta]$, we use the approximation

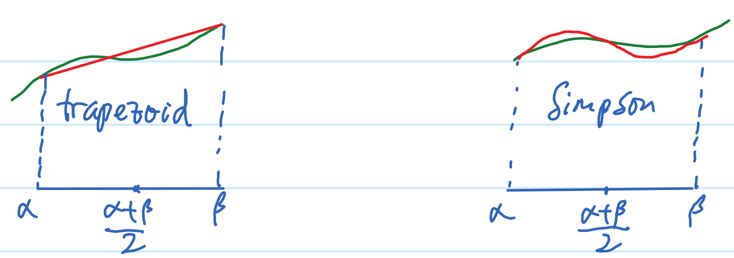
$$\int_\alpha^\beta f(x)\,dx \approx \underbrace{c_1}_{\text{weights}} f(\alpha) + c_2 f\left(\frac{\alpha+\beta}{2}\right) + c_3 f(\beta)$$

$e_n \sim 1/n \sim h$ | $e_n \sim 1/n \sim h$ | $e_n \sim 1/n^2 \sim h^2$



weights: 1 0 0 | 0 0 1 | 0 1 0

$e_n \sim 1/n^2 \sim h^2$ | $e_n \sim 1/n^4 \sim h^4$



weights: $1/2$ 0 $1/2$ | $1/6$ $2/3$ $1/6$

Simpson's method (by Simpson in 1700s) put weights $1/6, 2/3, 1/6$ on $\alpha, \frac{\alpha+\beta}{2}, \beta$ respectively. This method gives quite good convergence rate: $e_n = |I_n - I| \sim \frac{1}{n^4} \sim h^4$.

Where does the choice of weights $1/6, 2/3, 1/6$ come from?
It comes from approximating f on the interval $[\alpha, \beta]$ by the (quadratic) interpolation polynomial that fits three points

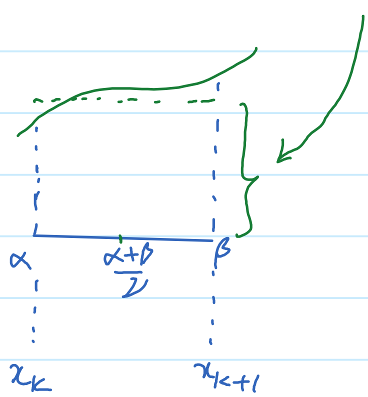
$$(\alpha, f(\alpha)), \quad \left(\frac{\alpha+\beta}{2}, f\left(\frac{\alpha+\beta}{2}\right)\right), \quad (\beta, f(\beta)).$$

Thus,

$$\int_\alpha^\beta f(x)\,dx \approx \int_\alpha^\beta P_2(x)\,dx$$

One can write an explicit formula of P_2 and then integrate it.
The result is

$$\underbrace{(\beta-\alpha)}_{\text{width}} \underbrace{\left(\frac{1}{6} f(\alpha) + \frac{2}{3} f\left(\frac{\alpha+\beta}{2}\right) + \frac{1}{6} f(\beta)\right)}_{\text{weighted height}}$$



Therefore,

$$I = \int_a^b f(x)\,dx = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} f(x)\,dx$$

$$\approx \underbrace{\sum_{k=0}^{n-1} h \left(\frac{1}{6} f(x_k) + \frac{2}{3} f\left(\frac{x_k + x_{k+1}}{2}\right) + \frac{1}{6} f(x_{k+1})\right)}_{S_n}$$

* Error estimate:

$$e_n^{(S)} = |S_n - I| \le \frac{\bar{M}(b-a)^5}{180} \frac{1}{n^4} \qquad \text{where } \bar{M} = \max_{[a,b]} |f^{(4)}(x)|.$$

We won't prove this inequality. It comes from the error estimate of interpolation polynomials.

Ex: $\quad I = \int_2^5 \frac{1}{x}\,dx$

$n = 6$

(a) Find I

(b) Find S_6.

(c) In order to make sure that $|S_n - I| < 10^{-4}$, how big should n be?

(a) $I = \log x \Big|_2^5 = \log(5/2) \approx 0.916290.$

(b)

x_0	x_1	x_2	x_3	x_4	x_5	x_6
2	2.5	3	3.5	4	4.5	5

$$h = \frac{5-2}{6} = 0.5$$

$$S_6 = \sum_{k=0}^{5} h\left(\frac{1}{6} f(x_k) + \frac{2}{3} f\left(\frac{x_k + x_{k+1}}{2}\right) + \frac{1}{6} f(x_{k+1})\right)$$

$$\begin{aligned}
&= 0.5\left(\tfrac{1}{6}\tfrac{1}{2} + \tfrac{2}{3}\tfrac{1}{2.25} + \tfrac{1}{6}\tfrac{1}{2.5}\right) \\
&+ 0.5\left(\tfrac{1}{6}\tfrac{1}{2.5} + \tfrac{2}{3}\tfrac{1}{2.75} + \tfrac{1}{6}\tfrac{1}{3}\right) \\
&+ 0.5\left(\tfrac{1}{6}\tfrac{1}{3} + \tfrac{2}{3}\tfrac{1}{3.25} + \tfrac{1}{6}\tfrac{1}{3.5}\right) \\
&+ 0.5\left(\tfrac{1}{6}\tfrac{1}{3.5} + \tfrac{2}{3}\tfrac{1}{3.75} + \tfrac{1}{6}\tfrac{1}{4}\right) \\
&+ 0.5\left(\tfrac{1}{6}\tfrac{1}{4} + \tfrac{2}{3}\tfrac{1}{4.25} + \tfrac{1}{6}\tfrac{1}{4.5}\right) \\
&+ 0.5\left(\tfrac{1}{6}\cdot\tfrac{1}{4.5} + \tfrac{2}{3}\cdot\tfrac{1}{4.75} + \tfrac{1}{6}\cdot\tfrac{1}{5}\right)
\end{aligned}$$

≈ 0.916298

(c) We have

$$|S_n - I| \le \frac{\bar{M}(5-2)^5}{180}\frac{1}{n^4} = \frac{27\bar{M}}{20}\frac{1}{n^4}$$

where $\bar{M} = \max_{[2,5]} |f^{(4)}(x)|$

Here $f(x) = 1/x$, $f'(x) = -1/x^2$, $f''(x) = 2/x^3$, $f^{(3)}(x) = -6/x^4$, $f^{(4)}(x) = 24/x^5$.

Thus,

$$\bar{M} = \max_{[2,5]} \frac{24}{x^5} \le \frac{24}{2^5} = \frac{3}{4}.$$

Hence,

$$e_n^{(S)} \le \frac{(27)\left(\frac{3}{4}\right)}{20} \frac{1}{n^4} = \frac{81}{80} \frac{1}{n^4}$$

To make sure that $e_n^{(S)} < 10^{-4}$, we simply require

$$\frac{81}{80} \frac{1}{n^4} < 10^{-4}$$

This is true when $n \geqslant 11$