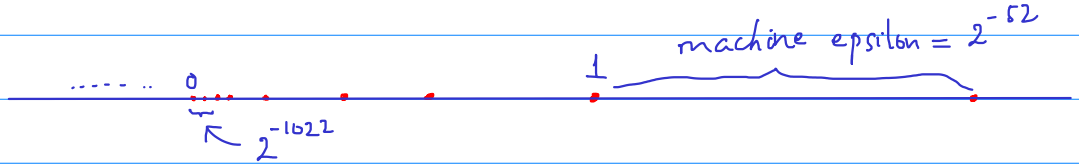
Lecture 7 (10/9/2019)

Some consequences of floating-point arithmetic error:

1) Loss of significant digits:

This can be seen by drawing a line and marking all the numbers that can be represented by IEEE-754 format.



$1 = (1.00\ldots0)_2 \times 2^0$

next number $= (1.00\ldots01)_2 \times 2^0$

machine $\varepsilon = (0.0\ldots01)_2 \times 2^0 = 2^{-52}$

The red dots represent the number that can be represented with exactness by the IEEE format. We see that these dots get sporadic when moving toward ∞ (or $-\infty$). Thus, if $x \gg y$ (x is much bigger than y) then $x+y$ will be rounded to x.

Loss of significant digits can also occur when multiplying too big number by a too small number. Some significant digits of the smaller number is lost due to rounding. Then this roundoff error is magnified by multiplication with the large number.

Ex:

$$\underbrace{x}_{\text{large}}\underbrace{(\sqrt{x+1}-\sqrt{x})}_{\text{small}} = \underbrace{\frac{x}{\sqrt{x+1}+\sqrt{x}}}_{\text{better for calculation}}$$

In Matlab, try two methods for $x = 10^{200}$.

2) Overflow and underflow:

This is caused by multiplying two too big numbers (overflow)

or two too small numbers (underflow).

Ex:

Compute $\sqrt{x^2+y^2}$ for $x = 10^{200}$, $y = 11^{200}$.

The size of $z = \sqrt{x^2+y^2}$ is of the same order as the size of x. But if one squares x and y, it will result in overflow (recall that the largest number that can be represented in IEEE 754 is about 10^{308}).

Instead, one can compute z another way:

$$z = x\sqrt{1+\left(\frac{y}{x}\right)^2}$$

3) Issue with choosing too small h in consideration of the limit as $h \to 0$:

Consider function $f(x) = x^2$. We want to evaluate $f'(1)$ on computer. By definition,

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(h+1)^2-1}{h}$$

Thus, $$f'(1) \approx \frac{(h+1)^2-1}{h}.$$

h would be considered as 0 in the IEEE 754 format if

$$h \le \underbrace{(0.00\ldots01)_2}_{51} \times 2^{-1022} = 2^{-1074}$$

$1+h$ would be considered as 1 if

$$1 \le 1+h \le \underbrace{(1.0\ldots01)_2}_{51} \times 2^0 = 1+2^{-52}$$

Therefore, if h is about 2^{-52} (or less), $1+h$ is considered as 1 and the numerator is equal to zero. The denominator is nonzero as long as $h > 2^{-1074}$.

$$2^{-52} \simeq 10^{-16}$$

One can check with Matlab that

$$\frac{(1+h)^2 - 1}{h} = 0 \quad \text{when } h = 10^{-16}$$

This error is caused by the nature of floating-point format. One should be aware of issues like this when programming an algorithm. For example, it is sufficient to select small h of order 10^{-8}, but not as small as 10^{-14}.