Lecture 8 (10/11/2019)

* Note on Problem 1 of HW2.

Consider the function $f(x) = xe^{-x^2}$. The motivation of this problem is the question: how to estimate f at certain values using only four basis arithmetic operations $(+, -, \times, /)$. For example, how to compute $f(2)$?

We do so by approaximating f by a polynomial. Let us write

$$f(x) = \underbrace{q_\ell(x)}_{\ell\text{-Taylor polynomial}} + \underbrace{E_\ell(x)}_{\text{error term}}.$$

It is difficult to estimate $E_\ell(x)$ using Lagrange's theorem because it is hard to find higher derivatives of f.

$$f'(x) = e^{-x^2} - x \cdot 2x e^{-x^2} = e^{-x^2}(1 - 2x^2)$$

$$\left.\begin{aligned} f''(x) &= \cdots \\ f'''(x) &= \cdots \end{aligned}\right\} \text{ more complicated}$$

Instead, we will take advantage of the Taylor expansion of the exponential function.

$$e^t = \underbrace{1 + \frac{t}{1!} + \frac{t^2}{2!} + \dots + \frac{t^n}{n!}}_{p_n(t)} + R_n(t)$$

Substitute $t = -x^2$:

$$e^{-x^2} = \sum_{k=0}^{n} \frac{(-x^2)^k}{k!} + R_n(-x^2)$$

Multiply both sides by x:

$$xe^{-x^2} = \underbrace{\sum_{k=0}^{n} \frac{(-1)^k}{k!} x^{2k+1}}_{q_{2n+1}(x)} + \underbrace{x R_n(-x^2)}_{E_{2n+1}(x)}$$

The problem of estimating $E_{2n+1}(x)$ becomes the problem of estimating $R_n(-x^2)$. Note that $t=-x^2$ varies between -4 and 0 as x varies between -2 and 2.

Now one can use Lagrange theorem to estimate the error term $R_n(t)$ of the function $g(t)=e^t$ for $-4\le t\le 0$.

* Continue with consequences of floating-point arithmetic error:

3) Accumulation of error:

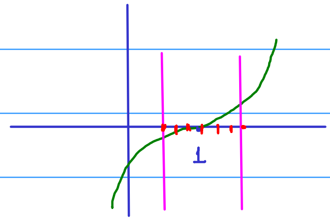
Consider the identity $(x-1)^3 = x^3-3x^2+3x-1$.

To compute the LHS, one needs 3 operations (1 subtraction, 2 multiplications).

To compute the RHS, one needs 8 (2 subtractions, 1 addition, 5 multiplications).

Computing the RHS typically gives larger error than LHS.

Ex:



Take 41 points around 1:

$x = 1-20h : h : 1+20h$

Then compute LHS:

$y = (x-1).\^{}3$

and RHS:

$z = x.\^{}3 - 3*x\^{}2 + 3*x - 1.$

Then plot y and z:

plot(x, y, '.b', x, z, '.r')

Blue dots for y, red dots for z. Then one see that if h is small ($\sim 10^{-8}$), y looks more like a cubic curve than z.

A lot of problems in real life can be modeled as solving an equation $f(x) = 0$.

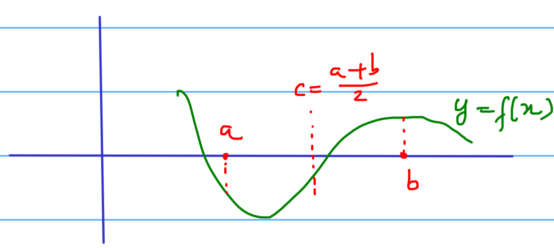
If a problem has more than one unknown, x would be a vector.

If a problem asks for a trajectory, density field, ... x would be a function.

Let us focus our attention to the simplest situation: where x is a real number.

Most of the time, the function f is complicated. It is not easy to find an exact value of x. In practice, most of the time we only need an approximate root. How to find an approximate value of x?

There are many strategies. One of the simplest and most natural one is the bisection method:



Suppose f is negative at a, and positive at b. From the property of continuous function (Intermediate Value theorem), we know that f must have a root in between a and b.

Our strategy is to shrink the interval $[a, b]$ to localize the root. In the figure above, $f(c)$ has the same sign as $f(a)$, but different sign from $f(b)$. We know that there is a root between c and b. Then examine the sign of $f(d)$, where d is the midpoint of $[c, b]$ and continue the procedure.

loop

$c = \frac{a+b}{2}$

If $f(a)f(c) < 0$ then view c as new b.

Otherwise, view c as new a.

We need a condition to break the loop. We can do so by introducing an error tolerence ε. The loop continues until $b-a \le \varepsilon$.