Lecture 9 (10/14/2019)

Want to find x such that $f(x) = 0$

Root finding by bisection method: the only requirements for this method to work are:

- f is continuous.
- one can find a, b such that $f(a)$ and $f(b)$ are of opposite sign.

Ex:

Find an approximate root of the equation $\cos x = x$.

First, we put this eq. in the form $f(x) = 0$. Here

$$f(x) = \cos x - x.$$

Next, we find a point where f is negative, and a point where f is positive.

$$f(0) = \cos 0 - 0 = 1 > 0$$

$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2} < 0.$$

One can take $a_0 = 0$ and $b_0 = \pi/2$. The interval $[a_0, b_0]$ is the initial interval.

$$c_0 = \frac{a_0 + b_0}{2} = \frac{\pi}{4}$$

$$f(c_0) = f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} - \frac{\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\pi}{4} < 0.$$

$$\begin{array}{ccc} a_0 & c_0 & b_0 \\ + & - & - \\ a_1 & b_1 & \end{array}$$

Thus, $a_1 = a_0$ and $b_1 = c_0 = \pi/4$. Then $c_1 = \frac{a_1 + b_1}{2} = \frac{\pi}{8}$.

Then find the sign of $f(c_1)$ ....

There are two ways to terminate this procedure:

1) Specify the number of iterations.

For example, we stop after 15 iterations and take

$$\underbrace{x_*}_{\text{exact root}} \approx \underbrace{x_0}_{\text{app. root}} = \frac{a_{15}+b_{15}}{2}$$

2) Repeat until the with of the interval $[a_n, b_n]$ is less than a prescribed error tolerance ε.

How to implement this in Matlab?

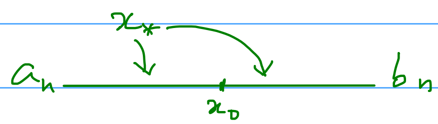
```
c= (a+b)/2
while ( b-a < ε)
      if   sign ( f(a)) * sign (f(c)) < 0
              b = c
      else
              a = c
      end
      c= (a+b)/2
end
```

* Error estimate:

Let $[a_0, b_0]$ be the initial interval. The length of $[a_1, b_1]$ is a half of the length of $[a_0, b_0]$. The length of $[a_2, b_2]$ is a half of the length of $[a_1, b_1]$, thus equal to a quarter of the length of $[a_0, b_0]$ ... In general,

$$b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$$

We know that x_* (the exact root) has to be somewhere in $[a_n, b_n]$. Thus, the difference between x_* and $x_0 = \frac{a_n+b_n}{2}$ is at most $\frac{b_n - a_n}{2}$.



Therefore, $|x_* - x_0| \le \frac{b_n - a_n}{2} = \frac{1}{2^{n+1}}(b_0 - a_0)$

For $|x_* - x_0|$ to be under a prescribed error ε, we only need to make sure that n is sufficiently big so that

$$\frac{1}{2^{n+1}}(b_0 - a_0) \le \varepsilon$$

This is equivalent to $2^{n+1} \ge \frac{b_0 - a_0}{\varepsilon}$.

This is equivalent to $n \ge \log_2\left(\frac{b_0 - a_0}{2}\right) - 1$.

See worksheet for example.