Solution to Problem 5, part (d):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{2x_n^3 + 2}{3x_n^2}$$

Put $\alpha = \sqrt[3]{2}$. Subtract $\sqrt[3]{2}$ from both sides :

$$x_{n+1} - \alpha = \frac{2x_n^3 + 2}{3x_n^2} - \alpha = \frac{2x_n^3 - 3\alpha x_n^2 + 2}{3x_n^2}$$

We want to factor as many factors $(x-\alpha)$ from the polynomial $2x^3 - 3\alpha x^2 + 2$ as possible. First, we see that $x=\alpha$ is a root of this polynomial, so $(x-\alpha)$ must be a factor of it. By balancing the powers of x on both sides,

$$2x^3 - 3\alpha x^2 + 2 = (x-\alpha)\left(2x^2 - \alpha x - \frac{2}{\alpha}\right)$$

Note that $x=\alpha$ is a root of $2x^2 - \alpha x - \frac{2}{\alpha}$. Thus, $(x-\alpha)$ is a factor of it.

$$2x^2 - \alpha x - \frac{2}{\alpha} = (x-\alpha)\left(2x + \frac{2}{\alpha^2}\right).$$

Put together:

$$2x^3 - 3\alpha x^2 + 2 = (x-\alpha)^2\left(2x + \frac{2}{\alpha^2}\right).$$

Therefore,

$$x_{n+1} - \alpha = \frac{(x_n-\alpha)^2\left(2x_n + \frac{2}{\alpha^2}\right)}{3x_n^2} \approx (x_n-\alpha)^2 \underbrace{\frac{2\alpha + \frac{2}{\alpha^2}}{3\alpha^2}}_{C}$$

The order of convergence is $p=2$.