Some review problems for Midterm

1. Consider the function $f(x) = xe^x$.

   (a) Find the degree n Taylor polynomial of f about $x_0 = 0$. Hint: use the Taylor expansion of e^x about 0.

   (b) Suppose we want to approximate xe^x by the polynomial $p_n(x)$ found above. For what values of n can we guarantee that the error of this approximation is at most $\epsilon = 10^{-6}$ for any $1 \le x \le 2$?

2. Convert 2.8 from decimal system to binary system. Round the result to 5 digits after the binary point.

3. Consider a floating-point system described as follows. A number x is represented approximately as $x \approx \sigma \cdot \bar{x} \cdot 2^e$ where

   - If $1 \le E \le 14$ then

$$\sigma = \begin{cases} 1 & \text{if } x \ge 0, \\ -1 & \text{if } x < 0, \end{cases}$$
$$e = E - 7,$$
$$\bar{x} = (1.a_1 a_2 a_3)_2 \quad \text{(rounding to truncate)}$$

   - If $E = 0$ then $e = -6$ and $\bar{x} = (0.a_1 a_2 a_3)_2$ (rounding to truncate).
   - If $E = 15$ then the bit sequence represents $\pm\infty$ (depending on the sign σ).

   (a) Represent the number 2.8 in this format.

   (b) Let $x = -(1.001)_2 \times 2^1$ and $y = (1.010)_2 \times 2^2$. Perform the operations $x + y$ and xy in this floating-point format.

4. Let $f(x) = x^3 - 3x + 1$.

   (a) Show that f has a root on the interval $(1, 2)$.

   (b) Use bisection method (by performing 4 iterations) to find a approximate root of f on this interval.

   (c) How many iterations are needed to obtain an approximate root with error at most $\epsilon = 10^{-8}$?

5. Suppose we want to compute approximately $\sqrt[3]{2}$ by using Newton's method for the function $f(x) = x^3 - 2$.

   (a) Write the iteration formula of Newton's method.

   (b) Pick $x_0 = 1$. Draw a picture that illustrates the Newton's method.

   (c) With the help of your calculator, find the approximate root after 4 iterations.

   (d) Find the order of convergence of x_n to $\sqrt[3]{2}$.

6. Suppose we want to compute approximately $\sqrt{2}$ by using fixed point method. Observe that it is a fixed point of $g(x) = \frac{2}{x}$.

   (a) Write the iteration formula of fixed point method.

   (b) For $x_0 = 1.5$, find x_1, x_2, x_3, x_4.

   (c) Draw a cobweb diagram that illustrates the fixed point method. Does the sequence x_n converge to $\sqrt{2}$?