# Worksheet

10/23/2019

**Name:** ______________________________

1. Consider a sequence $x_n$ given by $x_0 = 2$,

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{2x_n}.$$

Find an order of convergence of $x_n$ to 1.

$$x_{n+1} - 1 = \frac{x_n}{2} + \frac{1}{2x_n} - 1 = \frac{x_n^2 - 2x_n + 1}{2x_n} = \frac{(x_n - 1)^2}{2x_n} \qquad (*)$$

* Heuristic argument:

Because $x_n$ converges to 1, $x_n \approx 1$. Then

$$x_{n+1} - 1 \approx \frac{(x_n - 1)^2}{2}$$

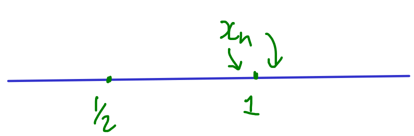
and thus

$$|x_{n+1} - 1| \lesssim \frac{|x_n - 1|^2}{2}$$

The order of convergence is $p = 2$.

* Rigorous argument:

Because $x_n$ converges to 1, $x_n > \frac{1}{2}$ for sufficiently large $n$.




Then $(*)$ implies

$$|x_{n+1} - 1| = \frac{|x_n - 1|^2}{2x_n} \leq \frac{|x_n - 1|^2}{2 \cdot \frac{1}{2}} = |x_n - 1|^2$$

We obtain $p = 2$.

2. Suppose we want to compute approximately $\sqrt{3}$ by using Newton's method for the function $f(x) = x^2 - 3$.

   - Write the iteration formula of Newton's method.
   - Choose $x_0 = 1$ as the initial iteration. Draw a picture that illustrates the Newton's method. From the picture, does the sequence $x_n$ converge?
   - Find the order of convergence of $x_n$ to $\sqrt{3}$.

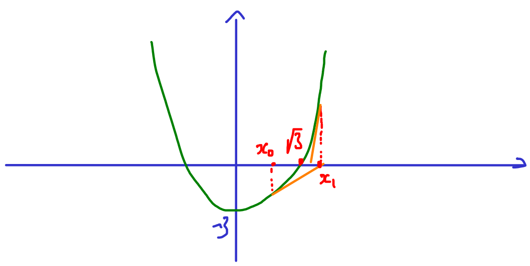
$f'(x) = 2x$

Iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 3}{2x_n} = \frac{x_n}{2} + \frac{3}{2x_n}$$

Then

$$x_{n+1} - \sqrt{3} = \frac{x_n}{2} + \frac{3}{2x_n} - \sqrt{3} = \frac{x_n^2 - 2\sqrt{3}x_n + 3}{2x_n} = \frac{(x_n - \sqrt{3})^2}{2x_n}$$




Because $x_n \to \sqrt{3}$, $x_n \approx \sqrt{3}$. Then

$$x_{n+1} - \sqrt{3} \approx \frac{(x_n - \sqrt{3})^2}{2\sqrt{3}}$$

Then

$$|x_{n+1} - \sqrt{3}| \lesssim \frac{|x_n - \sqrt{3}|^2}{2\sqrt{3}}$$

Order of convergence is $p = 2$.