Worksheet

10/25/2019

Name: ______________________________

1. Suppose we want to compute approximately $\sqrt{5}$ by using Newton's method for the function $f(x) = x^2 - 5$.

   - Write the iteration formula of Newton's method.
   - Choose $x_0 = 2$ as the initial iteration. Draw a picture that illustrates the Newton's method. From the picture, does the sequence x_n converge?
   - Find the order of convergence of x_n to $\sqrt{5}$.

$f'(x) = 2x$

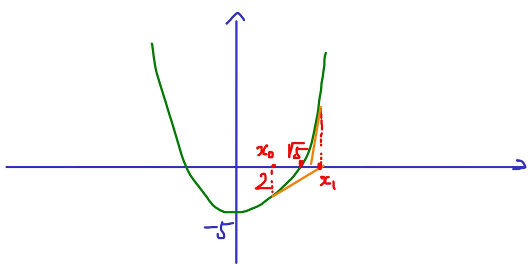
Iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{x_n^2 + 5}{2x_n}$$

Then

$$x_{n+1} - \sqrt{5} = \frac{x_n^2 + 5 - 2\sqrt{5}x_n}{2x_n}.$$

We want to factor the numerator. Notice that $\sqrt{5}$ is a root of the quadratic polynomial $r^2 + 5 - 2\sqrt{5}r$. Thus, $r - \sqrt{5}$ must be a factor of this polynomial. The other factor is also $r - \sqrt{5}$ because $\sqrt{5}$ is a double root. Therefore, $r^2 + 5 - 2\sqrt{5}r = (r - \sqrt{5})^2$.



We obtain

$$x_{n+1} - \sqrt{5} = \frac{(x_n - \sqrt{5})^2}{2x_n}$$

Because $x_n \to \sqrt{5}$, $x_n \approx \sqrt{5}$. Then

$$x_{n+1} - \sqrt{5} \approx \frac{(x_n - \sqrt{5})^2}{2\sqrt{5}}$$

Then

$$|x_{n+1} - \sqrt{5}| \lesssim \frac{|x_n - \sqrt{5}|^2}{2\sqrt{5}}$$

Order of convergence is $p = 2$.