Worksheet
9/30/2019

Name: ________________________________

Find the Taylor polynomial (of general degree), about $x_0 = 0$, of the following functions using the Σ notation.

(a) $\frac{1}{2-x}$
Bound the error in degree n approximation for $|x| \leq 1/2$.

Put $f(x) = \frac{1}{2-x}$

Then $f'(x) = \frac{1}{(2-x)^2}$, $f''(x) = 2\frac{1}{(2-x)^3}$,

$f'''(x) = 2 \cdot 3 \cdot \frac{1}{(2-x)^4}, \ldots, f^{(n)}(x) = 1 \cdot 2 \ldots n \cdot \frac{1}{(2-x)^{n+1}} = n! \frac{1}{(2-x)^{n+1}}$

Plug $x=0$: $\quad f^{(n)}(0) = \frac{n!}{2^{n+1}}$

Then $\quad f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x) = \underbrace{\sum_{k=0}^{n} \frac{x^k}{2^{k+1}}}_{P_n(x)} + R_n(x)$$

Estimate error: $\quad R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1}$

$$= \frac{(n+1)!}{(n+1)!} \frac{1}{(2-c_x)^{n+2}} x^{n+1} = \frac{x^{n+1}}{(2-c_x)^{n+2}}$$

Note that c_x is in between 0 and x. Because $|x| \leq 1/2$, both x and c_x are in between $-1/2$ and $1/2$. Then

$$|R_n(x)| = \frac{|x|^{n+1}}{(2-c_x)^{n+2}} \leq \frac{(1/2)^{n+1}}{(2-1/2)^{n+2}} = \frac{1}{2^{n+1}}\left(\frac{2}{3}\right)^{n+2}$$

$$|R_n(x)| \leq \frac{2}{3^{n+2}}$$

(b) $\frac{1}{2+3x}$

$$\frac{1}{2+3x} = \frac{1}{2} \frac{1}{1+\frac{3x}{2}} = \frac{1}{2}\frac{1}{1-u}$$

where $u = -\frac{3x}{2}$.

Using the identity

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$$

we get

$$\frac{1}{2+3x} = \frac{1}{2}\sum_{k=0}^{n}\left(-\frac{3x}{2}\right)^k + R_n(x) = \frac{1}{2}\sum_{k=0}^{n}\left(-\frac{3}{2}\right)^k x^k + R_n(x)$$

(c) $\frac{2}{(1-x)^2}$

Hint: $\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

Take derivative of both sides:

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1} = \sum_{\ell=0}^{\infty} (\ell+1) x^{\ell}$$

Thus,

$$\frac{1}{(1-x)^2} = \underbrace{\sum_{\ell=0}^{n} (\ell+1) x^{\ell}}_{p_n(x)} + R_n(x)$$

(d) $\arctan x$

Hint: $\arctan'(x) = \frac{1}{1+x^2}$

$$\frac{1}{1+x^2} = \frac{1}{1-u} \quad \text{where} \quad u = -x^2$$

$$= 1 + u + u^2 + u^3 + \cdots = \sum_{k=0}^{\infty} u^k = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

Taking anti-derivatives of both sides, we get:

$$\arctan x + C = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

Plug $x = 0$, get $C = 0$. Thus,

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = \underbrace{\sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{2k+1}}_{P_{2n-1}(x)} + R_{2n-1}(x)$$