Some review problems for Final

1. Consider function $f(z) = \frac{\operatorname{Log}(z+5)}{\sin z}$.

   (a) Determine all singular point(s) of f enclosed in the circle $C_4(0)$.

   (b) Are they isolated singularities? If so, which kind of isolated singularity are they (removable, pole, essential)?

   (c) Compute the residue of f at each of these singularities.

   (d) Evaluate the integral $\int_\gamma f(z)dz$ where γ is the circle $C_4(0)$ oriented counterclockwise.

2. Find the following limits.

   (a) $\lim_{z\to i} \frac{z^4-1}{z-i}$
   Hint: factor or use L'Hospital rule.

   (b) $\lim_{z\to 1+i} \frac{z^2+z-1-3i}{z^2-2z+2}$

   (c) $\lim_{z\to 0} \frac{z\operatorname{Re} z}{|z|}$

   (d) $\lim_{z\to\infty} \frac{z}{e^z}$

3. Find an antiderivative if exists of the following functions. Specify the domain of that antiderivative.

   (a) $f(z) = -2(xy+x) + i(x^2 - 2y - y^2)$
   Hint: write $F' = f$. Then use Cauchy-Riemann equations.

   (b) $f(z) = z \operatorname{Log} z$
   Hint: first regard z as real. Do integration by part. Double check formula by differentiation.

   (c) $f(z) = \bar{z}$

   (d) $f(z) = \frac{z-2}{z^2-z}$
   Hint: use partial fraction

   (e) $f(z) = \frac{1}{z^2+1}$
   Hint: use partial fraction

4. Compute the following integrals. *If you need to use a named theorem, make sure to specify it (Cauchy-Goursat, Fundamental theorem of Calculus, Cauchy's Integral formula, Cauchy's Residue theorem.)*

   (a) $\int_\gamma z^2 dz$
   where $\gamma(t) = (\sin t, t^2)$, $0 \le t \le \pi$.

   (b) $\int_\gamma \frac{1}{z} dz$
   where $\gamma(t) = e^{3it}$, $0 \le t \le 2\pi$. (Warning: γ is not a simple loop.)

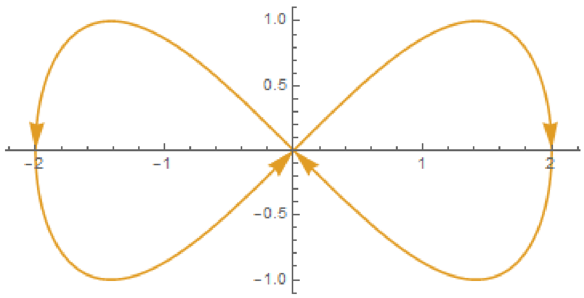
   (c) $\int_\gamma \bar{z} dz$
   where $\gamma(t) = (3t, t^2)$, $-1 \le t \le 2$.

   (d) $\int_\gamma \frac{e^z}{z(z-3)} dz$
   where γ is the unit circle oriented clockwise.

   (e) $\int_\gamma z^2 \sin(\frac{1}{z}) dz$
   where γ is the boundary of square with vertices at $\pm 1 \pm i$ negatively oriented.
   Hint: use Laurent series.

   (f) $\int_\gamma \frac{z+1}{(z-\frac{\pi}{2})^2 \sin z} dz$
   where γ is the circle $C_2(0)$ oriented counterclockwise.

   (g) $\int_\gamma \frac{z+2}{z^2-1} dz$
   where γ is the figure eight curve as in the picture.



Answer key

1. (a) $z = 0, -\pi, \pi$
   (b) Yes. Each is a pole of order 1 (single pole).
   (c) $\text{Res}[f;0] = \ln 5, \quad \text{Res}[f;\pi] = -\ln(5+\pi), \quad \text{Res}[f;-\pi] = -\ln(5-\pi)$
   (d) $2\pi i \ln \frac{5}{25-\pi^2}$

2. (a) $-4i$
   (b) $1 - \frac{3}{2}i$
   (c) 0
   (d) DNE

3. (a) $F(z) = u + iv$ where $u(x,y) = -x^2y - x^2 + y^2 + \frac{y^3}{3}$ and $v(x,y) = \frac{x^3}{3} - 2xy - xy^2$, valid on $\mathbb{C}$.
   (b) $F(z) = -\frac{z^2}{4} + \frac{z^2}{2}\,\text{Log}\, z$, valid on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$.
   (c) No antiderivatives
   (d) $F(z) = -\text{Log}(z-1) + 2\,\text{Log}\, z$, valid on $\mathbb{C}\backslash\mathbb{R}_{\leq 1}$
   (e) $F(z) = \frac{1}{2i}\text{Log}(z-i) - \frac{1}{2i}\text{Log}(z+i)$, valid on $\mathbb{C}$ minus to lines $\{t+i : t \leq 0\}$ and $\{t-i : t \leq 0\}$

4. (a) $-\frac{\pi^6}{3}i$
   (b) $6\pi i$
   (c) $21 + 9i$
   (d) $\frac{2\pi i}{3}$
   (e) $\frac{\pi i}{3}$
   (f) $2\pi i(1 + \frac{4}{\pi^2})$
   (g) $-4\pi i$

Formula to be provided on Final Exam

$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$

$\text{Log}(z+1) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$

Cauchy–Riemann equations:

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$

Cauchy's Integral formula:

$$\int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Cauchy's Residue formula:

$$\int_\gamma f(z)dz = 2\pi i(\mathrm{Res}\,[f;z_1] + \mathrm{Res}\,[f;z_2] + \cdots \mathrm{Res}\,[f;z_m])$$

If a is a pole of order n of function f then

$$\mathrm{Res}\,[f;a] = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]$$