HW 1 Solution.

#1

(a) $(2+i)\cdot(3+4i) = (6-4)+i(3+8) = 2+11i$

(b) $(1+2i)^4 = (1+2i)^2\cdot(1+2i)^2 = (-3+4i)\cdot(-3+4i) = -7-24i$

(c) $\dfrac{1+i}{2-3i} = \dfrac{(1+i)(2+3i)}{(2-3i)(2+3i)} = \dfrac{-1+5i}{16} = \dfrac{-1}{16}+\dfrac{5}{16}i$

(d) $\dfrac{a+i}{-a+i} = \dfrac{(a+i)(-a-i)}{(-a+i)(-a-i)} = \dfrac{-a^2+1}{a^2+1} + \dfrac{-2a}{a^2+1}i$

#2.

(a) $2i = 2e^{\frac{\pi}{2}i}$

(b) $1+i = \sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = \sqrt{2}\,e^{\frac{\pi}{4}i}$

(c) $-3+\sqrt{3}i = 3\sqrt{2}\cdot\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = 3\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right) = 3\sqrt{2}\,e^{\frac{3\pi}{4}i}$

(d) $-i = e^{-\frac{\pi}{2}i}$

(e) $\because\ 2-i = \sqrt{5}\left(\frac{2}{\sqrt{5}}-\frac{i}{\sqrt{5}}\right) = \sqrt{5}\cdot e^{i\phi}$, where $\phi\in\mathbb{R}$ s.t.: $\tan\phi = -\frac{1}{2}$

$\therefore\ (2-i)^2 = (\sqrt{5})^2 e^{i(2\phi)} = 5e^{i(2\phi)}$, $\phi = \tan^{-1}(-1/2)$.

(f) $|3-4i| = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25}$

(g) $\sqrt{5}-i = \sqrt{6}\left(\frac{\sqrt{5}}{\sqrt{6}}-\frac{i}{\sqrt{6}}\right) = \sqrt{6}\,e^{i\phi}$, $\phi = \tan^{-1}\left(\frac{-1}{\sqrt{5}}\right)$

(h) $\because\ \dfrac{1-i}{\sqrt{3}} = \dfrac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right) = \sqrt{\frac{2}{3}}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{\frac{2}{3}}\cdot e^{-\frac{\pi}{4}i}$

$\therefore\ \left(\dfrac{1-i}{\sqrt{3}}\right)^4 = \left(\sqrt{\frac{2}{3}}\right)^4 e^{-\pi i} = \frac{4}{9}e^{-i\pi}$ ∎

#3.

(a) $\sqrt{2}\,e^{i\frac{3\pi}{4}} = \sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = -1+i$

(b) $34\,e^{i\frac{\pi}{2}} = 34(0+i) = 34i$

(c) $e^{-i(250\pi)} = e^{-i\cdot 2\pi\cdot 125} = 1^{-125} = 1$

(d) $2e^{4\pi i} = 2(\cos 4\pi + i\sin 4\pi) = 2$

#4. (a) Use quadratic formula:

$$2z^2+2z+5=0$$

$$\Rightarrow z=\frac{-2\pm\sqrt{4-40}}{4}=-\frac{1}{2}\pm i\frac{3}{2}$$

(b) Use quadratic formula:

$$5z^2+4z+1=0$$

$$\Rightarrow z=\frac{-4\pm\sqrt{16-20}}{10}=-\frac{2}{5}\pm i\frac{1}{5}$$

(c) $z^2+2z+(1-i)=0$

Let $z=x+iy$, plug in the above equation, collect the real and imaginary part, we obtain

$$(x^2+2x+1-y^2)+i(2xy+2y-1)=0$$

so
$$\begin{cases}(x+1)^2-y^2=0 & (1)\\ y(x+1)=\frac{1}{2} & (2)\end{cases}$$

From (1), $y=\pm(x+1)$

(i) If $y=x+1$, then plug into (2) yields

$$(x+1)^2=\frac{1}{2}$$

$$\Rightarrow x=-1+\frac{1}{\sqrt{2}} \quad \text{or} \quad -1-\frac{1}{\sqrt{2}}$$

so that $y=\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

Hence, $z=(-1+\frac{1}{\sqrt{2}})+\frac{i}{\sqrt{2}}$ or $z=(-1-\frac{1}{\sqrt{2}})-\frac{i}{\sqrt{2}}$

(ii) If $y=-(x+1)$, then (2) gives $-(x+1)^2=\frac{1}{2}$, a contradiction!!

Thus, the solutions are

$$z_1=(-1+\frac{1}{\sqrt{2}})+\frac{i}{\sqrt{2}} \quad ; \quad z_2=(-1-\frac{1}{\sqrt{2}})-\frac{i}{\sqrt{2}}$$

(d) $z^4=z \Rightarrow z(z^3-1)=0$

$$\Rightarrow z=0 \quad \text{or} \quad z^3=1$$

$$\therefore z=0, \quad z=e^{i\frac{2k\pi}{3}}, \quad k=0,1,2$$

or $z=0,1,-\frac{1}{2}+i\frac{\sqrt{3}}{2},-\frac{1}{2}-i\frac{\sqrt{3}}{2}$

(e)

Let $w = z^2$. Then $z^4 - z^2 + 4 = 0 \Rightarrow w^2 - w + 4 = 0$

Quadratic formula : $w = \frac{1 \pm \sqrt{1-16}}{2} = \frac{1}{2} \pm i\frac{\sqrt{15}}{2}$

Write $w_1 = 2e^{i\theta_1}$, $\theta_1 = \tan^{-1}(\sqrt{15}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$

and $w_2 = 2e^{i\theta_2}$, $\theta_2 = \tan^{-1}(-\sqrt{15}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(i) $z^2 = w_1 \Rightarrow z = \sqrt{2} \cdot e^{i(\frac{\theta_1 + 2k\pi}{2})}$, $k = 0, 1$

(ii) $z^2 = w_2 \Rightarrow z = \sqrt{2} \cdot e^{i(\frac{\theta_2 + 2k\pi}{2})}$, $k = 0, 1$

Using half-angle formula and basic properties of exponential function, we have all the solutions are :

$$\left\{ \frac{\sqrt{5}}{2} + i\frac{\sqrt{3}}{2},\ -\frac{\sqrt{5}}{2} - i\frac{\sqrt{3}}{2},\ \frac{\sqrt{5}}{2} - i\frac{\sqrt{3}}{2},\ -\frac{\sqrt{5}}{2} + i\frac{\sqrt{3}}{2} \right\}$$

(f) Let $w = z^3$, then

$w^2 - w - 2 = 0 \quad \Rightarrow w = \frac{1 \pm \sqrt{1+8}}{2} = -1$ or 2

(i) $z^3 = -1 \Rightarrow z = e^{i(\frac{\pi + 2k\pi}{3})}$, $k = 0, 1, 2$

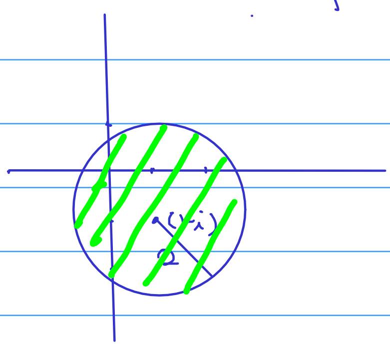
$= -1,\ \frac{1}{2} + i\frac{\sqrt{3}}{2},\ \frac{1}{2} - i\frac{\sqrt{3}}{2}$

(ii) $z^3 = 2 \Rightarrow z = \sqrt[3]{2}\, e^{i\frac{2k\pi}{3}}$, $k = 0, 1, 2$

$= \sqrt[3]{2},\ \sqrt[3]{2}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i),\ \sqrt[3]{2}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)$ ■

#5.

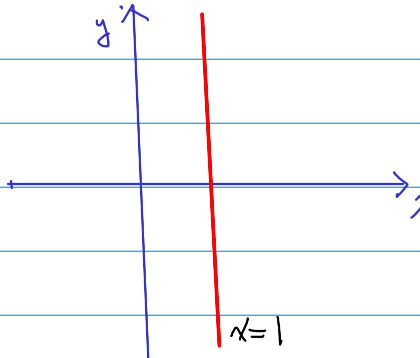
(a) $\{z \in \mathbb{C} : |z - 1 + i| = 2\}$



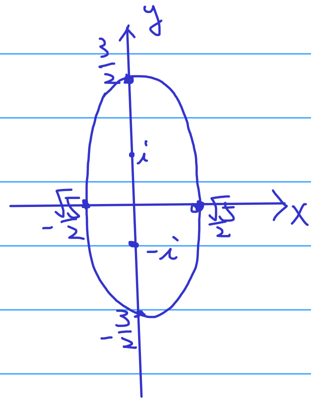
(c) $\{z \in \mathbb{C} : Re(z + 2 - 2i) = 3\}$

$= \{z \in \mathbb{C} : Re(z) + 2 = 3\}$

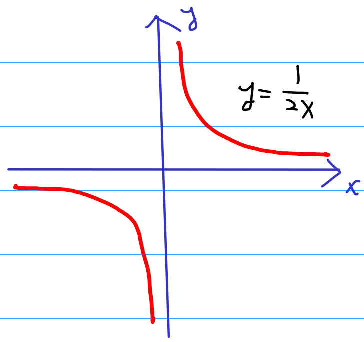
$= \{z \in \mathbb{C} : Re(z) = 1\}$



(d) $\{z\in\mathbb{C} : |z-i|+|z+i|=3\}$ ellipse with foci: $-i$, i

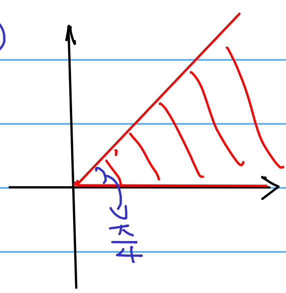


(h) $\{z\in\mathbb{C} : \mathrm{Im}(z^2)=1\} = \{z=x+iy\in\mathbb{C} : 2xy=1,\ x,y\in\mathbb{R}\}$

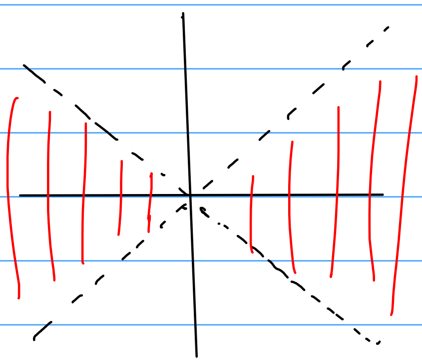


#6.

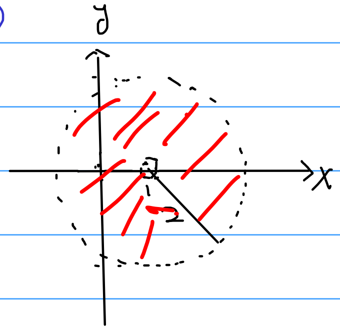
(a)



(b)



(c)



(d)