HW 2 Solutions

#1.

(a) $z+2\bar{z}=1$, set $z=x+iy$, $x,y\in\mathbb{R}$. Then

$3x-iy=1 \Rightarrow x=\frac{1}{3}$, $y=0$

So, $\underline{z=\frac{1}{3}}$

(b) Let $z=x+iy$ for $x,y\in\mathbb{R}$.

$0=2z^2+(i-1)z+5i = 2(x^2-y^2)+i(4xy)+(i-1)(x+iy)+5i$

$=(2x^2-x-2y^2-y)+i(4xy+x-y+5)$

$\Rightarrow \begin{cases} 2(x-\frac{1}{4})^2-2(y+\frac{1}{4})^2=0 & \text{— (1)} \\ 4xy+x-y+5=0 & \text{— (2)} \end{cases}$

(1) $\Rightarrow x=y+\frac{1}{2}$ or $x=-y$ ($x=y+\frac{1}{2}$ and (2) would yield complex y, not what we want)

Plug $x=-y$ into (2) $\Rightarrow 4y^2+2y-5=0 \Rightarrow y=-\frac{1}{4}\pm\frac{\sqrt{21}}{4}$

So $\begin{cases} z_1=\frac{1+\sqrt{21}}{4}+i\,\frac{-1-\sqrt{21}}{4} \\ z_2=\frac{1-\sqrt{21}}{4}+i\,\frac{-1+\sqrt{21}}{4} \end{cases}$

(c) Multiplying $z^2+2z^{-2}=-2$ by z^2 on both sides :

$z^4+2z^2+2=0$

Set $w=z^2 \Rightarrow w^2+2w+2=0$ and the quadratic formula gives

$w=-1\pm i$

- For $w=-1+i=\sqrt{2}\,e^{i\frac{3\pi}{4}}=z^2$

  $\therefore\ z=\underline{2^{\frac{1}{4}}e^{i(\frac{3\pi}{8})}}$, $\underline{2^{\frac{1}{4}}e^{i(\frac{11}{8}\pi)}}$

- For $w=-1-i=\sqrt{2}\,e^{-i\frac{3\pi}{4}}=z^2$

  $z=\underline{2^{\frac{1}{4}}e^{-i\frac{3\pi}{8}}}$, $\underline{2^{\frac{1}{4}}e^{i\frac{5}{8}\pi}}$

(d) Note that $z^3+iz^2+7z-5i=(z-i)\cdot(z^2+2iz+5)=0$

Our goal now is to solve $z^2+2iz+5=0$ for z.

Again, letting $z=x+iy$ and substitute into $z^2+2iz+5=0$:

$(x^2-y^2-2y+5)+i(2xy+2x)=0$

$$\begin{cases} x^2-y^2-2y+5=0 \\ 2x(y+1)=0 \end{cases} \Rightarrow x=0,\ y=-1\pm\sqrt{6}$$

So,

$z_1 = i(-1+\sqrt{6})$, $z_2 = i(-1-\sqrt{6})$, $z_3 = i$

are all the solutions. ■

#2. (a) $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \sqrt{2}\, e^{i\frac{\pi}{4}}$, $\mathrm{Arg}(1+i)=\frac{\pi}{4}$

$\therefore \sqrt[3]{1+i} = \left\{ \sqrt[6]{2}\cdot \mathrm{cis}\left(\frac{\pi}{12}+\frac{2k\pi}{3}\right) : k=0,1,2 \right\}$

$= \left\{ \sqrt[6]{2}\,\mathrm{cis}\left(\frac{\pi}{12}\right),\ \sqrt[6]{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right),\ \sqrt[6]{2}\,\mathrm{cis}\left(\frac{17\pi}{12}\right) \right\}$

(b) $\because i = 1\cdot e^{i\frac{\pi}{2}}$, $\mathrm{Arg}(i)=\frac{\pi}{2}$

$\therefore \sqrt[4]{i} = \left\{ 1\cdot \mathrm{cis}\left(\frac{\pi}{8}+\frac{2k\pi}{4}\right) : k=0,1,2,3 \right\}$

$= \left\{ \mathrm{cis}\left(\frac{\pi}{8}\right),\ \mathrm{cis}\left(\frac{5\pi}{8}\right),\ \mathrm{cis}\left(\frac{9\pi}{8}\right),\ \mathrm{cis}\left(\frac{13}{8}\pi\right) \right\}$

(c) $\because -1 = 1\cdot e^{i\pi}$, $\mathrm{Arg}(-1)=\pi$

$\therefore \sqrt[5]{-1} = \left\{ 1\cdot \mathrm{cis}\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right) : k=0,1,2,3,4 \right\}$

$= \left\{ \mathrm{cis}\left(\frac{\pi}{5}\right),\ \mathrm{cis}\left(\frac{3\pi}{5}\right),\ \mathrm{cis}(\pi),\ \mathrm{cis}\left(\frac{7\pi}{5}\right),\ \mathrm{cis}\left(\frac{9}{5}\pi\right) \right\}$

(d) $\because 1+2i = \sqrt{5}\left(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i\right) = \sqrt{5}\cdot \mathrm{cis}(\theta)$, $\theta = \arctan(2) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\therefore (1+2i)^{\frac{3}{2}} = \left\{ 5^{\frac{3}{4}}\cdot \mathrm{cis}\left(\frac{3}{2}(\theta+2k\pi)\right) : k=0,1 \right\}$

$= \left\{ 5^{\frac{3}{4}}\left(\cos\frac{3\theta}{2}+i\sin\frac{3\theta}{2}\right),\ 5^{\frac{3}{4}}\cdot\left(\cos\left(\frac{3\theta}{2}+\pi\right)+i\sin\left(\frac{3\theta}{2}+\pi\right)\right) \right\}$

#3.

From de Moivre's formula we know that

$$[\cos(x)+i\sin(x)]^3 = \cos(3x)+i\sin(3x). \quad (*)$$

expand the LHS of $(*)$:

$$[\cos(x)+i\sin(x)]^3 = [\cos^3(x)-3\sin^2(x)\cos(x)] + i\cdot[3\cos^2(x)\sin(x)-\sin^3(x)] \quad (**)$$

Comparing the real part and imaginary part of $(*)$ and $(**)$, we obtain:

$$\begin{cases} \cos(3x) = \cos^3(x)-3\sin^2(x)\cos(x) \\ \sin(3x) = 3\cos^2(x)\sin(x)-\sin^3(x) \end{cases}$$

■

#4. In this problem, we let $z = x+iy$ and $w = u+iv$, for $x,y,u,v\in\mathbb{R}$

(a). True.

$$z+w = (x+u)+i(y+v) \Rightarrow \overline{z+w} = (x+u)-i(y+v)$$
$$= (x-iy)+(u-iv)$$
$$= \bar{z}+\bar{w}$$

(b) True.

$$z\cdot w = (xu-yv)+i(xv+yu) \Rightarrow \overline{zw} = (xu-yv)-i(xv+yu)$$

and $\bar{z}.\bar{w} = (x-iy).(u-iv) = (xu-yv)+i(-xv-yu) = (xu-yv)-i(xv+yu)$

$$= \overline{zw}$$

(c) False.

Let $z=-1$, $w=1$ Then $|z+w| = 0 \neq 2 = |z|+|w|$.

(d) True.

$$|zw|^2 = (xu-yv)^2+(xv+yu)^2 = (x^2u^2-2xuyv+y^2v^2)+(x^2v^2+2xvyu+y^2u^2)$$
$$= x^2(u^2+v^2)+y^2(u^2+v^2)$$
$$= (x^2+y^2)\cdot(u^2+v^2)$$
$$= |z|\cdot|w|$$

or, alternatively, let $z = r_1e^{i\theta_1}$, $w = r_2e^{i\theta_2}$. then $z\cdot w = r_1r_2\cdot \text{cis}(\theta_1+\theta_2)$

$\therefore |zw| = r_1r_2\cdot|\text{cis}(\theta_1+\theta_2)| = r_1\cdot r_2\cdot 1 = |z|\cdot|w|$.

(e) True.

Assume $\text{Arg}(z)=\theta_1$, $\text{Arg}(w)=\theta_2$

Then $\text{Arg}(zw) = (\theta_1+\theta_2) \pmod{2\pi}$

$= \text{Arg}(z) + \text{Arg}(w) \pmod{2\pi}$

(f) False.

Let $z=-1=w$. Then $\text{Arg}(z) = \pi = \text{Arg}(w)$

and $\text{Arg}(zw) = \text{Arg}(1) = 0 \neq 2\pi = \text{Arg}(z) + \text{Arg}(w)$.

(g) False.

Let $z=1$, $w=i$. Then $\arg(z) = 0 + 2k\pi$, $\arg(w) = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

but $\arg(z+w) = \arg(1+i) = \frac{\pi}{4} + 2k\pi$

$\therefore \arg(z+w) \neq \arg(z) + \arg(w) \pmod{2\pi}$