Homework 2

Due 4/17/2019

1. Find all complex solutions (written in either standard form or polar form) of the following equations.

   (a) $z + 2\bar{z} = 1$

   (b) $2z^2 + (i-1)z + 5i = 0$

   (c) $z^2 + 2z^{-2} = -2$

   (d) $z^3 + iz^2 + 7z - 5i = 0$
   Hint: note that $z = i$ is a root. Then factor the left hand side by long division.

2. Find all complex roots:

   (a) $\sqrt[3]{i+1}$

   (b) $\sqrt[4]{i}$

   (c) $\sqrt[5]{-1}$

   (d) $\sqrt{(1+2i)^3}$

3. Use de Moivre's formula $\operatorname{cis}^n(x) = \operatorname{cis}(nx)$ to express $\cos 3x$ and $\sin 3x$ in terms of $\cos x$ and $\sin x$.

4. Determine if each of the following statements is true. If it is, prove it. If it is not, give a counterexample.

   (a) $\overline{z+w} = \bar{z} + \bar{w}$

   (b) $\overline{zw} = \bar{z}\bar{w}$

   (c) $|z+w| = |z| + |w|$

   (d) $|zw| = |z||w|$

   (e) $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$ (modulo 2π)

   (f) $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$

   (g) $\arg(z+w) = \arg z + \arg w$ (modulo 2π)

The purpose of Problems 5, 6, 7 is to visualize complex-variable functions. You will need Mathematica. Supplemental material on how to use basic plotting commands on Mathematica can be found on the course website.

We know that the graph of a function $f : D \subset \mathbb{R} \to \mathbb{R}$ is a curve (or "1 dimensional manifold") on the plane $\mathbb{R}^2$. And the graph of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a surface (or "2-dimensional manifold") in the space $\mathbb{R}^3$. How about the graph of a function $f : D \subset \mathbb{C} \to \mathbb{C}$? We know that $\mathbb{C}$ can be represented geometrically as the plane $\mathbb{R}^2$. Then the graph of such function f would be a "3-dimensional manifold" in $\mathbb{R}^4$, which is not possible to visualize as a whole. However, there are a number of methods to visualize particular aspects of this graph. (This practice is somewhat similar to drawing many 2D pictures of a 3D object.) Below are two useful methods.

- Method 1: plot the real part, imaginary part, modulus, (principal) argument: $\operatorname{Re} f(z)$, $\operatorname{Im} f(z)$, $|f(z)|$, $\operatorname{Arg} f(z)$. For example, the function $f(z) = z$ (the identity map) can considered as $f(x,y) = (x,y)$. Then $\operatorname{Re} f(z) = x$, $\operatorname{Im} f(z) = y$, etc. One can plot $\operatorname{Re} f(z)$ using the command **Plot3D** in Mathematica:

  ```
  Plot3D[x, {x,-1,1}, {y,-1,1}]
  ```

- Method 2: plot the image of curves. In this respect, one is viewing f as a geometric transformation on the plane. For example, the function $f(z) = z^2$ can be considered as $f(x, y) = (x^2 - y^2, 2xy)$. The line $y = 1$ has parametric equation $(x, y) = (t, 1)$. Its image under f is a curve with parametric equation $(u, v) = (t^2 - 1, 2t)$. One can plot this curve using the command **ParametricPlot** in Mathematica:

```
ParametricPlot[{t^2-1,2t}, {t,-3,3}]
```

In the following problems, make sure to write the Mathematica code you use, give explanation and some comments on the graph.

5. Consider the function $f(z) = 1/z$. Plot $\operatorname{Re} f(z)$, $\operatorname{Im} f(z)$, $|f(z)|$ and $\operatorname{Arg} f(z)$. Hint: the function Arg is a built-in function in Mathematica. (Type, for example, Arg[2+3*I] to test.)

6. Consider that function $f(z) = z^2 - z$.

   (a) Plot the images under f of three curves: $x = 1$, $y = 1$, the circle centered at $(1, 1)$ with radius 2 on the same graph. Hint: write the parametric equation for each curve and use ParametricPlot.

   (b) What do you notice about the angles made between the original curves and the angles made between their images?

7. The multi-valued function $\arg z$ can be visualized by following the below steps:

   (a) Write the complex number z in polar form. What are $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ in terms of r and θ?

   (b) Plot the surface $x = x(r, \theta)$, $y = y(r, \theta)$, $w = \theta$ using the command **ParametricPlot3D**. (For better visualization, choose the range of θ to be wide, for example from 0 to 10π, and the range of r to be relatively wider than 2π, for example from 0 to 10.)