HW 3 Solution

#1.

(a) $4^i = e^{i\,\mathrm{Log}4} = e^{i\ln 4} = \cos(\ln 4) + i\sin(\ln 4)$.

(b) $(1+i\sqrt{3})^{\frac{i}{2}} = e^{\frac{i}{2}\mathrm{Log}(1+i\sqrt{3})} = \exp\left(\frac{i}{2}\cdot\left(\ln 2 + i\frac{\pi}{3}\right)\right) = \exp\left(-\frac{\pi}{6} + i\frac{\ln 2}{2}\right)$

$= e^{-\pi/6}\cos\left(\frac{\ln 2}{2}\right) + i\, e^{-\pi/6}\sin\left(\frac{\ln 2}{2}\right)$

(c) $(-1)^{\sqrt{2}} = e^{\sqrt{2}\cdot\mathrm{Log}(-1)} = e^{\sqrt{2}\cdot i\pi} = \cos(\sqrt{2}\pi) + i\sin(\sqrt{2}\pi)$.

(d) $(i-1)^{2i+3} = e^{(2i+3)\mathrm{Log}(i-1)} = e^{(2i+3)\cdot\left(\ln\sqrt{2} + i\frac{3\pi}{4}\right)} = \exp\left(3\ln\sqrt{2} - \frac{3}{2}\pi\right)\cdot\cos\left(2\ln\sqrt{2} + \frac{\pi}{4}\right)$

$+ i\exp\left(3\ln\sqrt{2} - \frac{3}{2}\pi\right)\cdot\sin\left(2\ln\sqrt{2} + \frac{\pi}{4}\right)$.

(e) $\because \sin(i) = \dfrac{e^{i\cdot i} - e^{-i\cdot i}}{2i} = \left(\dfrac{e - e^{-1}}{2}\right)\cdot i = i\cdot\sinh(1)$

$\therefore e^{\sin(i)} = \cos(\sinh(1)) + i\sin(\sinh(1))$.

(f) $\cos(-2+i) = \dfrac{e^{-2i-1} + e^{1+2i}}{2} = \dfrac{e+e^{-1}}{2}\cdot\cos(2) + i\cdot\dfrac{e-e^{-1}}{2}\cdot\sin(2)$

$= (\cos(2)\cdot\cosh(1)) + i(\sin(2)\cdot\sinh(1))$

(g) $\sin\left(\frac{\pi}{4}+i\right) = \dfrac{e^{-1+\frac{\pi}{4}i} - e^{1-\frac{\pi}{4}i}}{2i} = \dfrac{e^{-1}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) - e\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)}{2i} = \frac{1}{\sqrt{2}}\cosh(1) + i\frac{1}{\sqrt{2}}\sinh(1)$.

(h) $\tan\left(\frac{\pi}{2}+\frac{i}{2}\right) = \dfrac{\sin\left(\frac{\pi}{2}+\frac{i}{2}\right)}{\cos\left(\frac{\pi}{2}+\frac{i}{2}\right)} = \dfrac{\sin\left(\frac{\pi}{2}\right)\cosh\left(\frac{1}{2}\right) + i\cos\left(\frac{\pi}{2}\right)\sinh\left(\frac{1}{2}\right)}{\cos\left(\frac{\pi}{2}\right)\cosh\left(\frac{1}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\sinh\left(\frac{1}{2}\right)} = i\left[\dfrac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)}\right]$

(i) $\cosh\left(1 - i\frac{\pi}{4}\right) = \dfrac{e\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) + e^{-1}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}{2} = \frac{1}{\sqrt{2}}\cosh(1) - i\frac{1}{\sqrt{2}}\sinh(1)$

(j) $\sinh(1+i\pi) = \dfrac{e(-1+0i) - e^{-1}(-1+0i)}{2i} = i\cdot\sinh(1)$.

#2

(a) $\sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = 2\,\mathrm{cis}\left(-\frac{\pi}{6} + k2\pi\right)$. $k\in\mathbb{Z}$

$\Rightarrow \log(\sqrt{3}-i) = \ln(2) + i\left(-\frac{\pi}{6} + 2k\pi\right)$, $k = 0, \pm1, \pm2, \ldots$

(b) $-ie = e\cdot\mathrm{cis}\left(-\frac{\pi}{2} + 2k\pi\right)$, $k\in\mathbb{Z}$

$\Rightarrow \log(-ie) = \ln(e) + i\left(-\frac{\pi}{2} + 2k\pi\right)$, $k\in\mathbb{Z}$

$= 1 + i\left(-\frac{\pi}{2} + 2k\pi\right)$, $k\in\mathbb{Z}$.

(c) $\sin^{-1}(1+i) = -i\log\left[-1+i+\sqrt{1-2i}\right]$

and $\sqrt{1-2i} = \sqrt{5}\ \mathrm{cis}\left(\frac{\theta}{2}+k\pi\right)$, $\theta = \tan^{-1}(-2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $k=0,1$

(1) $k=0$, $\sin^{-1}(1+i) = -i\log\left[-1+\sqrt{\frac{1+1/\sqrt{5}}{2}} + i\left(1-\sqrt{\frac{1-1/\sqrt{5}}{2}}\right)\right]$

(2) $k=1$, $\sin^{-1}(1+i) = -i\log\left[-1-\sqrt{\frac{1+1/\sqrt{5}}{2}} + i\left(1+\sqrt{\frac{1-1/\sqrt{5}}{2}}\right)\right]$

#3 (a) False.

Let $z=i$. Then $\sqrt{z} = \sqrt{i} = \mathrm{cis}\left(\frac{\pi}{4}\right)$. (principal branch $(-\pi,\pi]$).

so $i\sqrt{z} = \mathrm{cis}\left(\frac{\pi}{2}\right)\cdot \mathrm{cis}\left(\frac{\pi}{4}\right) = \mathrm{cis}\left(\frac{3\pi}{4}\right)$

On the other hand,

$$\sqrt{-z} = \sqrt{-i} = \mathrm{cis}\left(-\frac{\pi}{4}\right)$$

so $\sqrt{-z} \neq i\sqrt{z}$.

(b) True.

$$\sin^2 z + \cos^2 z = \frac{e^{i2z} - 2 + e^{-i2z}}{-4} + \frac{e^{i2z} + 2 + e^{-i2z}}{4} = \frac{4}{4} = 1$$

(c) False.

Let $z = i$. Then

$$\sin(i) = \frac{e^{-1} - e}{2i} \; ; \quad \cos(i) = \frac{e^{-1} + e}{2}$$

so $$|\sin(i)|^2 + |\cos(i)|^2 = \frac{e^{-2} + e^2}{2} \neq 1.$$

(d) True.

By definition.

(e) True.

Let $z = x + iy$. Then

$$\sin(\pi - z) = \frac{e^{y + i(\pi - x)} - e^{-y - i(\pi - x)}}{2i}$$

$$= \frac{e^{y}\sin(\pi - x) + e^{-y}\sin(\pi - x)}{2} + \frac{e^{y}\cos(\pi - x) - e^{-y}\cos(\pi - x)}{2i}$$

$$= \frac{\sin(x)(e^{y} + e^{-y})}{2} + \frac{\cos(x)(e^{-y} - e^{y})}{2i}$$

$$= \sin(z).$$

(f) True.

Let $z = x + iy$. Then

$$e^{\bar{z}} = e^{x}(\cos y - i \sin y) = \overline{e^{x}\cos y + i e^{x}\sin y} = \overline{e^{z}} \;\blacksquare$$

#4. We use the principal branch : $(-\pi, \pi]$.

$$\therefore f(z) = z^{1/4} = e^{\frac{1}{4}\log z}$$

$$= e^{\frac{1}{4}(\ln|z| + i(\operatorname{Arg} z + 2k\pi))}$$

$$= |z|^{\frac{1}{4}} \cdot \operatorname{cis}\left(\frac{\operatorname{Arg} z}{4} + \frac{k\pi}{2}\right), \quad k = 0, 1, 2, 3. \text{ to be determined.}$$

Plug in $z = 1$, $-i = f(1) = 1 \cdot \operatorname{cis}(0 + \frac{k\pi}{2}) \Rightarrow k = 3$

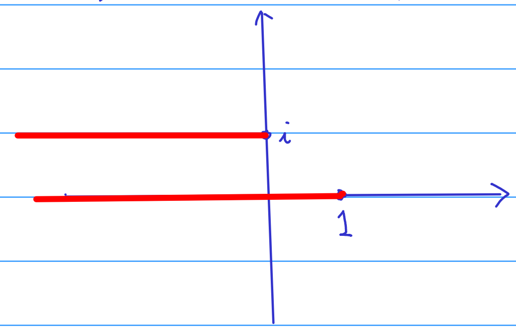
So, $$f(4i) = |4i|^{\frac{1}{4}} \cdot \operatorname{cis}\left(\frac{\pi/2}{4} + \frac{3\pi}{2}\right)$$

$$= 4^{\frac{1}{4}} \cdot \operatorname{cis}\left(\frac{13}{8}\pi\right) = \sqrt{2}\cos\left(\frac{13}{8}\pi\right) + i\sqrt{2}\sin\left(\frac{13}{8}\pi\right). \;\blacksquare$$

#5.

Remove all z's with $z-1 \le 0$ and $z-i \le 0$

- $z-1 \le 0 \Rightarrow$ if $z = a+ib$, $a,b \in \mathbb{R}$, then $b=0$, $a \le 1$
- $z-i \le 0 \Rightarrow$ if $z = a+ib$, $a,b \in \mathbb{R}$, then $b=1$, $a \le 0$



Domain $D = \mathbb{C} \setminus \left(\{x+i \in \mathbb{C} : x \in \mathbb{R},\ x \le 0\} \cup \{x+iy \in \mathbb{C} : x,y \in \mathbb{R},\ x \le 1,\ y = 0\} \right)$

#6.

(a) principal branch : $(-\pi, \pi]$, branch cut : $\mathbb{C} \setminus \mathbb{R}_{\le 0}$. branch point of $\log$: 0.

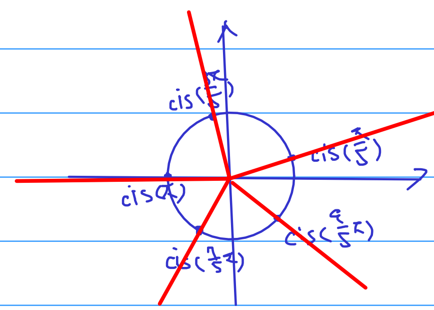
cut out those z's with $z^5 \le 0$.

Assume $z^5 = a \le 0$, $a = re^{i\pi}$, $r \ge 0$

$$\Rightarrow z = r^{\frac{1}{5}} e^{i\left(\frac{\pi + 2k\pi}{5}\right)}, \quad k = 0, 1, 2, 3, 4.$$

$$= \left\{ r^{\frac{1}{5}} e^{i\frac{\pi}{5}},\ r^{\frac{1}{5}} e^{i\frac{3\pi}{5}},\ r^{\frac{1}{5}} e^{i\pi},\ r^{\frac{1}{5}} e^{i\frac{7\pi}{5}},\ r^{\frac{1}{5}} e^{i\frac{9\pi}{5}} \right\}$$

Domain of $\log(z^5)$ on the branch $(-\pi, \pi]$:



$$D = \mathbb{C} \setminus \left(\left\{ re^{i\theta} \in \mathbb{C} : \theta = \frac{(2k+1)\pi}{5},\ k = 0, 1, 2, 3, 4,\ r \ge 0 \right\} \right)$$

(b) principal branch : $(-\pi, \pi]$

branch cut : $\mathbb{C} \setminus \mathbb{R}_{\le 0}$.

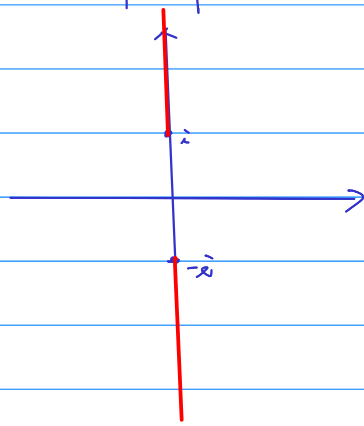
Remove z : $z^2 + 1 \le 0$.

Since $z^2 + 1 = (z+i)(z-i)$, assume $z = x+iy$, plug in $(z+i)(z-i) \le 0$

obtain

$$(x^2 - y^2 + 1) + i(2xy) \le 0 \Rightarrow x \cdot y = 0, \quad x^2 \le y^2 - 1 \Rightarrow x = 0, \ y^2 \ge 1.$$

Domain of $\log(z^2+1)$ on the principal branch $(-\pi, \pi]$:



$$D = \mathbb{C} \setminus \{ iy \in \mathbb{C} : y \in \mathbb{R},\ |y| \geq 1 \}$$