Homework 3

Due 4/25/2019

Problems 1, 2, 3 are computational. *Make sure to write down every computational step.* You can double check your answers using Mathematica, for example,

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N[Sin[1 - I]]
1.29846 - 0.634964 I
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1. Write the following complex numbers in standard form. Use the principal branch of the logarithm if necessary.

   (a) 4^i
   (b) $(1+i\sqrt{3})^{i/2}$
   (c) $(-1)^{\sqrt{2}}$
   (d) $(i-1)^{2i+3}$
   (e) $e^{\sin i}$
   (f) $\cos(-2+i)$
   (g) $\sin\left(\frac{\pi+4i}{4}\right)$
   (h) $\tan\left(\frac{\pi+i}{2}\right)$
   (i) $\cosh\left(\frac{4-i\pi}{4}\right)$
   (j) $\sinh(1+i\pi)$

2. Find all complex values of the following:

   (a) $\log(\sqrt{3}-i)$
   (b) $\log(-ie)$
   (c) $\arcsin(1+i)$

3. Determine if each of the following statements is true. If it is, prove it. If it is not, give a counterexample.

   (a) $\sqrt{-z} = i\sqrt{z}$
   (principal branch being used)
   (b) $\cos^2 z + \sin^2 z = 1$
   (c) $|\cos z|^2 + |\sin z|^2 = 1$
   (d) $\sin(z+2\pi) = \sin z$
   (e) $\sin(\pi - z) = \sin z$
   (f) $e^{\overline{z}} = \overline{e^z}$

4. Let $f(z)$ be a branch of the multi-valued function $z^{1/4}$ such that $f(1) = -i$. Find $f(4i)$.

5. Let $f(z) = \sqrt{z-1}\sqrt[3]{z-i}$ where each root function is defined using the principal logarithm (whose domain is $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$). Find the domain of f.

6. To each multi-valued function in Part (a) and (b), define a *continuous single-valued* branch for it using the following procedure:

   - Choose a branch for the logarithm. Recall that this amounts to choosing a branch for the argument function $\arg z$. Restricting the argument to any interval of length 2π, for example, $(-\pi,\pi]$, $[0,2\pi)$, $[-\pi/4, 7\pi/4)$, etc would each give a branch.
   - Cut the complex plane: each interval chosen for the argument corresponds to a curve to be removed from the complex plane. For example, if the interval $[0,2\pi)$ is chosen, the curve to remove is $\{z : \text{Arg}z = 0\} \cup \{0\}$.
   - Determine the domain of this single-valued function. That is, the set of all z's such that the expression inside the logarithm does not lie on the branch cut. (Geometry is useful.)
   - Testing: pick two points in the domain of the above function $f(z)$ (say, $z_1 = 2i+1$ and $z_2 = -2+i$ if possible). Compute $f(z_1)$ and $f(z_2)$.

(Note that Step 2 and 3 would not be necessary if the continuity requirement is removed.)

(a) $\log(z^5)$

(b) $\log(z^2 + 1)$

The purpose of the following problem is to use Mathematica to visualize some mapping properties of the function $f(z) = \sin z$. *Make sure to write the Mathematica code you use, give explanation and some comments on the graph.* Similar properties of the exponential function e^z are addressed in the supplemental material "**Mapping properties via Mathematica**" posted on the course website.

From Problem 2, we know that $f(z+2\pi) = f(z)$ and $f(\pi - z) = f(z)$. To consider some mapping properties of $f(z)$, we restrict the domain of f to the infinite vertical strip $[-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\infty, \infty)$.

7. (a) Draw the image under f of the rectangle $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-1, 1]$.

   (b) Draw the images under f some grid lines (horizontal and vertical). Based on the picture, what are the images of the line $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$?

   (c) What region does f map the vertical strip $[-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\infty, \infty)$ onto ? Is it also a one-to-one mapping?

   (d) What region does f map the open vertical strip $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\infty, \infty)$ onto ? Is it also a one-to-one mapping?