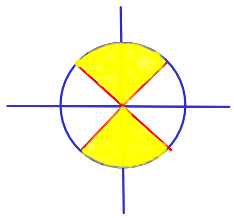
HW 4 Solution

#1. (a)



(b) $\mathring{G} = \{ z \in \mathbb{C} : |z| < 2 \text{ and } Re(z^2) < 1 \}$.

(c) $\partial G = \{ z \in \mathbb{C} : |z| = 2 \text{ and } Re(z) \in [-\sqrt{2}, \sqrt{2}] \} \cup \{ z \in \mathbb{C} : Re(z) = Im(z), \text{ and } Re(z) \in [-\sqrt{2}, \sqrt{2}] \}$
$\cup \{ z \in \mathbb{C} : Re(z) = -Im(z) \text{ and } Re(z) \in [-\sqrt{2}, \sqrt{2}] \}$

(d) G is not open, since it contains one of the boundary set $\{ z \in \mathbb{C} : Re(z) = Im(z), -\sqrt{2} \le Re(z) \le \sqrt{2} \}$.

G is not closed, since it does not contain the boundary points on $\{ z \in \mathbb{C} : |z| = 2 \text{ and } -\sqrt{2} \le Re(z) \le \sqrt{2} \}$.

Conclusion: G is neither closed nor open.

#2. In this problem, we denote by R, the region of continuity.

(a) $R = \mathbb{C}$

proof: Given $\varepsilon > 0$, pick $\delta = \varepsilon$. s.t. if for all $z, w \in \mathbb{C}$ with $|z-w| < \delta$,
then $|f(z) - f(w)| = |\bar{z} - \bar{w}| = |\overline{z-w}| = |z-w| < \delta = \varepsilon$

(b) $R = \mathbb{C}$

proof: Given $\varepsilon > 0$, choose $\delta = \varepsilon$ s.t. if $\forall z, w \in \mathbb{C}$ and $|z-w| < \delta$

$\Rightarrow |f(z) - f(w)| = \big| |z| - |w| \big| \le |z-w|$ (triangle inequality)

$< \delta = \varepsilon$.

(c) $R = \mathbb{C}$

proof: It is sufficient to show e^z is continuous on $\mathbb{C}$.
Fix $z \in \mathbb{C}$. Let $\{w_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{C}$ s.t. $w_j \to z$ as $j \to \infty$

Let $w_j = a_j + ib_j$, $z = a + ib$

Then $|e^{w_j} - e^z| = |e^{a_j} e^{ib_j} - e^a \cdot e^{ib}| \le$

$$|e^{a_j}e^{ib_j} - e^{a}\cdot e^{ib_j}| + |e^{a}\cdot e^{ib_j} - e^{a}\cdot e^{ib}|$$

$$\leq |e^{a_j} - e^{a}|\cdot 1 + e^{a}\cdot |e^{ib_j} - e^{ib}|$$

$$\leq |e^{a_j} - e^{a}| + 2e^{a}\left(|\cos(b_j) - \cos(b)| + |\sin(b_j) - \sin(b)|\right)$$

$\to 0$ as $j \to \infty$

since $a_j \to a$, $b_j \to b$ as $j \to \infty$ and the continuity of $\cos$, $\sin$ and e^x.

(d) $R = \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq -1\}$

Note that $(z+1)^{\frac{1}{2}} = e^{\frac{1}{2}\mathrm{Log}(z+1)}$ and since $\mathrm{Log}(z+1)$ is continuous at the points outside the set $\{z \in \mathbb{R} : z \leq -1\}$ or $\mathbb{R}_{\leq -1}$, so $(z+1)^{\frac{1}{2}}$ is continuous on R.

(e) $R = \mathbb{C} \setminus \left(\{z = a + i \in \mathbb{C} : a \in \mathbb{R}, a \leq 0\} \cup \{z = a - i \in \mathbb{C} : a \in \mathbb{R}, a \leq 0\}\right)$

#3. (a) $\gamma(t) = (1+i) + 3e^{it}$, $0 \leq t \leq 2\pi$

(b) $\gamma(t) = (-1-i) + t(1+3i)$, $0 \leq t \leq 1$

(c) $\gamma(t) = (1-2i) + t(1+3i)$, $t \in \mathbb{R}$

(d) $\gamma(t) = (-1+i) + 2e^{-it}$, $0 \leq t \leq \pi$.

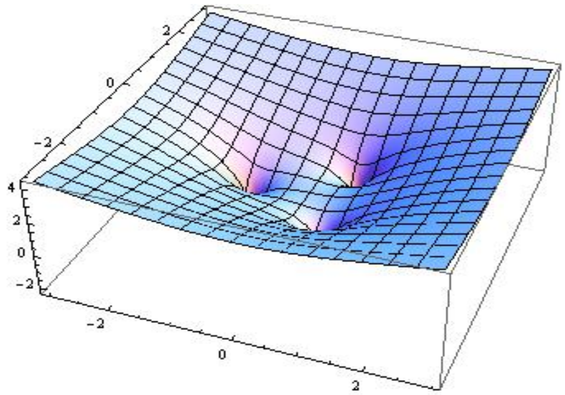
#5.

(a) $g(z) = \log(z^3+1) = \ln|z^3+1| + i\arg(z^3+1)$

$\therefore \mathrm{Re}\, g(z) = \ln|z^3+1|$

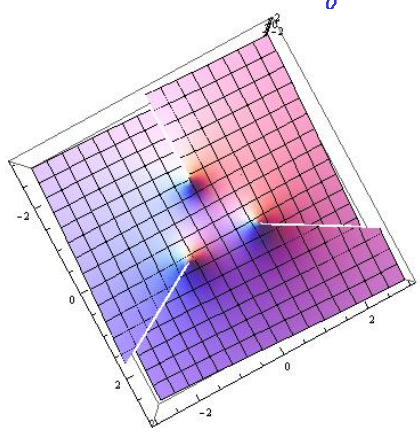
$\mathrm{Im}\, g(z) = \arg(z^3+1)$

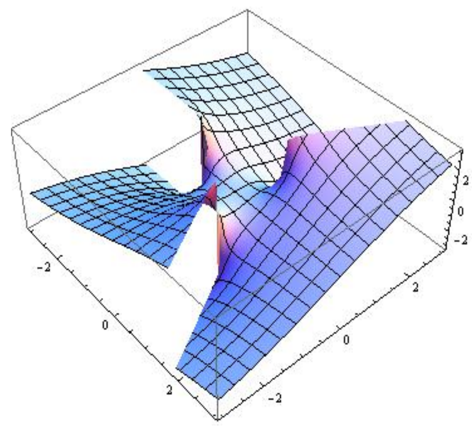
(b)



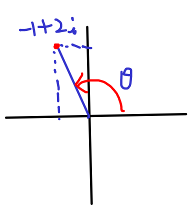
(c) & (d)

$$F(z) = \mathrm{Arg}(z^3+1) \quad ; \quad f(z) = F(z) + k2\pi,\ k \in \mathbb{Z}.$$





$$F(1+i) = \mathrm{Arg}((1+i)^3+1) = \mathrm{Arg}(-1+2i) = \tan^{-1}(-2) + \pi \approx 2.0344$$



(f) Branch points come from the cubic root of $z^3+1=0$

or $z^3 = -1 = e^{i(\pi+2k\pi)}$

$\therefore\ z = e^{i\left(\frac{\pi+2k\pi}{3}\right)}$, $k = -1, 0, 1$

so, branch points are: $z_1 = e^{i\frac{\pi}{3}} = \underline{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

$z_2 = e^{i\pi} = \underline{-1}$

$z_3 = e^{-i\frac{\pi}{3}} = \underline{\frac{1}{2} - i\frac{\sqrt{3}}{2}}$

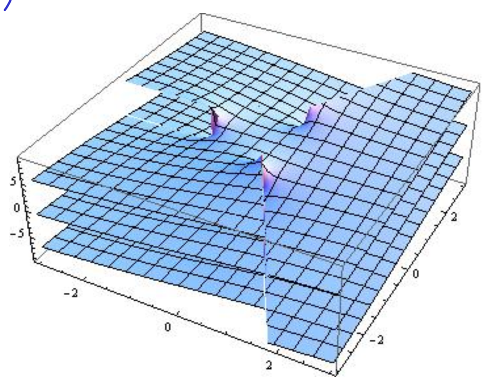
(g) branch cuts:

$$C_1 = \{ r e^{i\frac{\pi}{3}} \in \mathbb{C} : r \in \mathbb{R}, r \geq 1 \}$$

$$C_2 = \{ r e^{i\pi} \in \mathbb{C} : r \in \mathbb{R}, r \geq 1 \}$$

$$C_3 = \{ r e^{-i\frac{\pi}{3}} \in \mathbb{C} : r \in \mathbb{R}, r \geq 1 \}.$$

(h)



*#4

(a) $\left.\begin{aligned} \frac{\partial f}{\partial x} &= -e^{-x} \cdot e^{-iy} \\ \frac{\partial f}{\partial y} &= -i e^{-x} \cdot e^{-iy} \end{aligned}\right\}$ exist and continuous on $\mathbb{C}$

and $-i \cdot \frac{\partial f}{\partial y} = -e^{-x} \cdot e^{-iy} = \frac{\partial f}{\partial x} \rightarrow$ Cauchy-Riemann eqn. holds.

Therefore, f is entire and $f'(z) = \frac{\partial f}{\partial x}(z) = -e^{-x} \cdot e^{-iy} = -e^{-z}$, if $z = x+iy$

(b) $\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2 + iy^2 \\ \frac{\partial f}{\partial y} &= i \cdot 2xy \end{aligned}\right\}$ both exist and continuous on $\mathbb{C}$

but $-i \frac{\partial f}{\partial y} = 2xy \neq 2 + iy^2 = \frac{\partial f}{\partial x}$ for all $x, y \in \mathbb{R}$.

i.e, f is nowhere differentiable according to Thm 2.13 (a).

(h) Let $z = x+iy$. Then $f(z) = z\,\mathrm{Im}\,z = (x+iy)\cdot y = xy + iy^2$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= y \\ \frac{\partial f}{\partial y} &= x + i2y \end{aligned}\right\} \text{ both exist and conti. on } \mathbb{C}.$$

$$-i\frac{\partial f}{\partial y} = 2y - ix$$

The C-R eqn holds when $2y - ix = y \Rightarrow x = y = 0$

i.e., f is diff. @ $(0,0)$ but nowhere holomorphic. $\blacksquare$