HW 5 Solution

#1. (a)

$$\because\ z^3+i=0 \Rightarrow z^3=-i \Rightarrow z=\left\{\frac{\sqrt3}{2}-\frac{i}{2},\ i,\ -\frac{\sqrt3}{2}-\frac{i}{2}\right\}$$

$$\therefore\ \frac{z^3+i}{z-i}=\frac{(z-i)\left(z-\left(\frac{\sqrt3}{2}-\frac{i}{2}\right)\right)\cdot\left(z-\left(-\frac{\sqrt3}{2}-\frac{i}{2}\right)\right)}{z-i}=\left[z-\left(\frac{\sqrt3}{2}-\frac{i}{2}\right)\right]\left[z+\frac{\sqrt3}{2}+\frac{i}{2}\right]$$

$$\Rightarrow \lim_{z\to i}\frac{z^3+i}{z-i}=\lim_{z\to i}\left[z-\left(\frac{\sqrt3}{2}-\frac{i}{2}\right)\right]\left[z+\frac{\sqrt3}{2}+\frac{i}{2}\right]=\underline{\left(-\frac{\sqrt3}{2}+\frac{i}{2}\right)\cdot\left(\frac{\sqrt3}{2}+\frac{3}{2}i\right)}$$

(b)

$$\lim_{z\to0}\frac{\mathrm{Log}(z+i)-\mathrm{Log}(i)}{z}=\lim_{z\to0}\frac{\mathrm{Log}(z+i)-\mathrm{Log}(0+i)}{z-0}=\frac{d}{dz}\mathrm{Log}(z+i)\Big|_{z=0}$$

$$=\frac{1}{z+i}\Big|_{z=0}=\underline{\frac{1}{i}}\ \text{ or }\ \underline{-i}$$

(c)

$$\lim_{z\to0}\frac{e^z-1}{\mathrm{Log}(z+1)}=\lim_{z\to0}\left[\frac{e^z-1}{z}\cdot\frac{z}{\mathrm{Log}(z+1)}\right]=\left[\lim_{z\to0}\frac{e^z-e^0}{z-0}\right]\cdot\left[\lim_{z\to0}\frac{1}{\frac{\mathrm{Log}(z+1)-\mathrm{Log}(1)}{z-0}}\right]$$

$$=\frac{d}{dz}e^z\Big|_{z=0}\cdot\left[\frac{d}{dz}\mathrm{Log}(z+1)\Big|_{z=0}\right]^{-1}=1\cdot1=\underline{1}$$

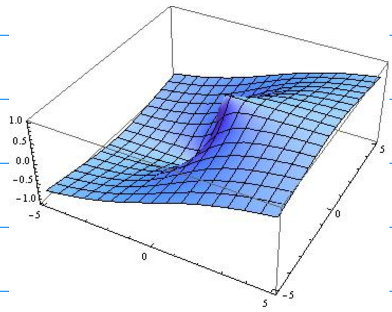
(d)

$$\lim_{z\to\infty}z\cdot\sin\left(\frac{1}{z}\right)\ \overset{w=\frac{1}{z}}{=\!=}\ \lim_{w\to0}\frac{\sin(w)}{w}=\underline{1}$$

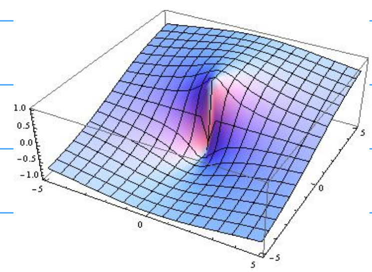
#2. (a) Let $z=x+iy$.

Then
$$\frac{z}{|z|}=\frac{x+iy}{\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}}+i\,\frac{y}{\sqrt{x^2+y^2}}:=u(x,y)+i\,v(x,y)$$

(b) Plot3D[x/Sqrt[x^2+y^2], {x,-5,5}, {y,-5,5}]; Plot3D[y/Sqrt[x^2+y^2], {x,-5,5}, {y,-5,5}]



Graph of u.



Graph of v.

(c)

- $$\lim_{\substack{(x,0)\to(0,0)\\ x<0}} \left(\frac{x}{\sqrt{x^2}} + i\,0\right) = \lim_{\substack{x\to 0\\ x<0}} \frac{-|x|}{|x|} = \boxed{-1}$$

- $$\lim_{\substack{x\to 0\\ x>0}} \left(\frac{x}{\sqrt{x^2}} + i\,0\right) = \lim_{\substack{x\to 0\\ x>0}} \frac{x}{x} = \boxed{1}$$

- $$\lim_{\substack{y\to 0\\ y<0}} \left(0 + i\,\frac{y}{\sqrt{y^2}}\right) = \lim_{\substack{y\to 0\\ y<0}} \left(i\,\frac{-|y|}{|y|}\right) = \boxed{-i}$$

- $$\lim_{\substack{y\to 0\\ y>0}} \left(0 + i\,\frac{y}{\sqrt{y^2}}\right) = 1\cdot i = \boxed{i}$$

(d)

- $$\lim_{\substack{x\to\infty\\ y=0}} \frac{x}{\sqrt{x^2}} = \lim_{x\to\infty} \frac{x}{|x|} = \boxed{1}$$

- $$\lim_{\substack{y\to\infty\\ x=0}} i\,\frac{y}{\sqrt{y^2}} = i\cdot \lim_{y\to\infty} \frac{y}{|y|} = \boxed{i}$$

#3.

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$$
$$= \ln\sqrt{x^2+y^2} + i\,\mathrm{Arg}(z), \quad \text{if } z = x+iy \in \mathbb{C}\setminus\mathbb{R}_{\le 0}$$

where
$$\mathrm{Arg}(z) = \begin{cases} \arctan\left(\frac{y}{x}\right), & x>0 \\ \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right), & y>0 \\ -\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right), & y<0. \end{cases} \quad (\ast)$$

Denote $\mathrm{Log}(z) = u(x,y) + i\,v(x,y)$, where $u = \ln\sqrt{x^2+y^2}$, v is defined as in $(\ast)$

- $$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$$

- $$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

- $$\frac{\partial v}{\partial x} = \begin{cases} \frac{-y}{x^2+y^2}, & x>0 \\ \frac{-y}{x^2+y^2}, & y>0 \end{cases}$$

$$\left\{ \quad \frac{|y|}{x^2+y^2}, \quad y<0. \right.$$

$$\cdot \ \frac{\partial v}{\partial y} = \begin{cases} \dfrac{x}{x^2+y^2}, & x>0 \\ \dfrac{x}{x^2+y^2}, & y>0 \\ \dfrac{x}{x^2+y^2}, & y<0 \end{cases}$$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous on the set $\mathbb{R}^2 \setminus \{(x,0) : x \le 0\}$.

and note that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(x,0) : x \le 0\}$$

Therefore, $\mathrm{Log}(z)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ by the Cauchy-Riemann Theorem ∎

#4. First notice that $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ is simply connected.

So,
$$\frac{d}{dz}F(z) = \frac{d}{dz}(z\,\mathrm{Log}\,z - z)$$
$$= \frac{d}{dz}(z) \cdot \mathrm{Log}\,z + z \cdot \frac{d}{dz}(\mathrm{Log}\,z) - \frac{d}{dz}(z) \quad \text{differentiation rules.}$$
$$= 1 \cdot \mathrm{Log}\,z + z \cdot \frac{1}{z} - 1$$
$$= \mathrm{Log}\,z \ ∎$$

#5. (a) $f(z) = \mathrm{Log}\,z + \mathrm{Log}(iz - i)$

- $\mathrm{Log}\,z$ is holomorphic in the region $\mathbb{C} \setminus \mathbb{R}_{\le 0}$
- $\mathrm{Log}(iz-i)$ is holomorphic in the region where $iz - i \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$

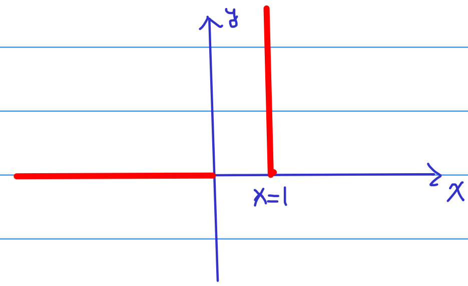
$\Rightarrow iz - i = a \in \mathbb{R}_{\le 0}$

$\Rightarrow i(x+iy) - i = a \in \mathbb{R}_{\le 0} \quad (z = x+iy)$

$\Rightarrow -y + i(x-1) = a \in \mathbb{R}_{\le 0}$

$\Rightarrow x = 1, \ y \ge 0$

Hence, $f(z)$ is holomorphic in the region depicted below: (out side of the red lines)



Formally, $R = \mathbb{C} \setminus \left(\{ z = x+iy \in \mathbb{C} : x \le 0, y = 0 \} \cup \{ z = x+iy \in \mathbb{C} : x = 1, y \ge 0 \} \right)$

(b) Following similar method as we find antiderivative of real variable functions, it is easy to check that the following function is an anti-derivative of f.

$$F(z) = z \operatorname{Log} z - 2z + z \operatorname{Log}(iz - i) - \operatorname{Log}(z-1) + C.$$

where $c \in \mathbb{C}$ is a constant.

#6. The parametrization of $C_2(0)$: $\gamma(t) = 2e^{it}$, $t \in [0, \pi]$

$\Rightarrow \gamma(t) = x(t) + iy(t) := 2\cos(t) + i\, 2\sin(t)$, $t \in [0, \pi]$

(a) $f(z) = z + \bar{z} = 2x$

$$\therefore \int_\gamma f(z)\,dz = \int_0^\pi f(\gamma(t)) \cdot \gamma'(t)\,dt = \int_0^\pi 2x(t) \cdot (x'(t) + i y'(t))\,dt$$

$$= \int_0^\pi [-8\sin(t) \cdot \cos(t) + i\, 8\cos^2 t]\,dt$$

$$= 4\cos^2(t)\Big|_0^\pi + 4i \cdot \left[t + \frac{\sin(2t)}{2}\right]\Big|_0^\pi = \underline{4\pi i}$$

(b) $f(z) = z^2 - 2z + 3 = (x^2 - y^2 - 2x + 3) + i(2xy - 2y)$

so
$$\int_\gamma f(z)\,dz = \int_0^\pi [(4\cos^2 t - 4\sin^2 t - 4\cos t + 3) + i(8\cos t \sin t - 4\sin t)] \cdot (-2\sin t + i\, 2\cos t)\,dt$$

$$= \int_0^\pi [2\sin t - 32\cos^2 t \sin t + 16\cos t \sin t]\,dt + i \cdot \int_0^\pi [14\cos t - 32\sin^2 t \cos t - 8\cos(2t)]\,dt$$

$$= \left(-2\cos t + \frac{32}{3}\cos^3 t + 8\sin^2 t\right)\Big|_{t=0}^{t=\pi} + i\left(14\sin t - \frac{32}{3}\sin^3 t - 4\sin(2t)\right)\Big|_{t=0}^{t=\pi}$$

$$= \underline{-\frac{52}{3}}$$

(c) $f(z) = xy = 4\cos t \cdot \sin t$

$$\Rightarrow \int_{\gamma} f(z)\, dz = \int_0^{\pi} \left(8\cos^2 t \cdot \sin t + i\, 8 \sin^2 t \cos t\right) dt$$

$$= \left(-\frac{8}{3}\cos^3 t\right)\Big|_{t=0}^{t=\pi} + i\left(\frac{8}{3}\sin^3 t\right)\Big|_{t=0}^{t=\pi} = \frac{16}{3}$$

(d) $f(z) = z^{-4} \Rightarrow f(\gamma(t)) = (2e^{it})^{-4} = 2^{-4} \cdot e^{-i4t}$. $t \in [0, \pi]$

$$\therefore \int_{\gamma} f(z)\, dz = \int_0^{\pi} \frac{1}{16}\left(\cos(4t) - i\sin(4t)\right)\cdot\left(-2\sin t + i\, 2\cos t\right) dt$$

$$= \frac{1}{8}\int_0^{\pi} \left[\sin(4t)\cos(t) - \cos(4t)\sin(t)\right] dt + i \cdot \frac{1}{8}\int_0^{\pi} \left[\cos(4t)\cos(t) + \sin(4t)\sin(t)\right] dt$$

$$= \frac{1}{8}\int_0^{\pi} \sin(3t)\, dt + i\frac{1}{8}\int_0^{\pi} \cos(3t)\, dt$$

$$= \frac{1}{12}$$