Homework 5

Due 5/16/2019

1. Evaluate the following limits. You can use the command **Limit** in Mathematica to double check your results. Try the example: `Limit[1/(z + I), z -> Infinity]`

   (a) $\lim_{z\to i} \frac{z^3+i}{z-i}$

   (b) $\lim_{z\to 0} \frac{\operatorname{Log}(z+i)-\operatorname{Log} i}{z}$

   (c) $\lim_{z\to 0} \frac{e^z-1}{\operatorname{Log}(z+1)}$

   (d) $\lim_{z\to\infty} z\sin\left(\frac{1}{z}\right)$
   Hint: change variable $w = 1/z$.

2. Consider the function $f(z) = z/|z|$.

   (a) Write $f(z)$ in complex standard form $f = u + iv$. In other words, determine $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$.

   (b) Use Mathematica to plot u and v. Hint: use command **Plot3D**.

   (c) Find the limit of $f(z)$ as z approaches 0 along each of the following paths:
   - the negative side of the real axis,
   - the positive side of the real axis,
   - the negative side of the imaginary axis,
   - the positive side of the imaginary axis.

   (d) Find the limit of $f(z)$ as z approaches ∞ along each of the following paths:
   - the positive side of the real axis,
   - the positive side of the imaginary axis.

3. The principal argument is given by

$$\operatorname{Arg} z = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0, \\ \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y > 0, \\ -\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y < 0, \end{cases}$$

   where $z = x + iy$. Use Cauchy–Riemann theorem to verify that the function $f(z) = \operatorname{Log} z$ is holomorphic on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$.

4. Verify that an antiderivative of $f(z) = \operatorname{Log} z$ on the region $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ is $F(z) = z\operatorname{Log} z - z$.

5. Consider the function $f(z) = \operatorname{Log} z + \operatorname{Log}(iz - i)$.

   (a) Determine the region where $f(z)$ is holomorphic.

   (b) Determine all antiderivatives of $f(z)$ on this region.

6. (Similar to Problem 4.5 on page 69 of the textbook) *Use the definition* of complex integration to integrate the following functions over the *upper semicircle* $C_2(0)$ oriented counter-clockwise.

   (a) $f(z) = z + \bar{z}$

   (b) $f(z) = z^2 - 2z + 3$

   (c) $f(z) = xy$

   (d) $f(z) = \frac{1}{z^4}$
   Hint: use de Moivre's formula.