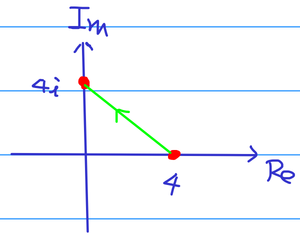
HW 6 Solution



#1. $\gamma(t) = (1-t)\cdot 4 + t\cdot(4i), \quad t \in [0,1]$

$= 4 + t\cdot(4i-4), \quad t\in[0,1]$

(a) $f(z) = \frac{z+1}{z} = 1 + \frac{1}{z}$ and $F(z) = z + \operatorname{Log} z$ is an antiderivative of f

$$\Rightarrow \int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0))$$

$$= 4i - 4 + \operatorname{Log}(4i) - \operatorname{Log}(4) \quad [\text{by Theorem 4.11 of the textbook}]$$

$$= \underline{-4 + i\left(\tfrac{\pi}{2} + 4\right)}$$

(b) $f(z) = \frac{1}{z^2+z} = \frac{1}{z} - \frac{1}{z+1}$; $F(z) = \operatorname{Log}(z) - \operatorname{Log}(z+1)$ is an antiderivative of f

$$\Rightarrow \int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0)) \qquad (\text{Theorem 4.11})$$

$$= \operatorname{Log}(4i) - \operatorname{Log}(1+4i) - \operatorname{Log}(4) + \operatorname{Log}(5)$$

$$= \underline{\ln\left(\tfrac{5}{\sqrt{17}}\right) + i\left(\tfrac{\pi}{2} - \tan^{-1}4\right)}$$

(c) $f(z) = z^{-1/2}$; $F(z) = 2z^{1/2}$ (principal branch)

$$\Rightarrow \int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0))$$

$$= 2(4i)^{1/2} - 2\sqrt{4} = \underline{4e^{i\frac{\pi}{4}} - 4}$$

(d) $f(z) = \sin^2(z) = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{e^{i(2z)} - 2 + e^{-i(2z)}}{-4} = \frac{1}{2} - \frac{1}{2}\cos(2z)$

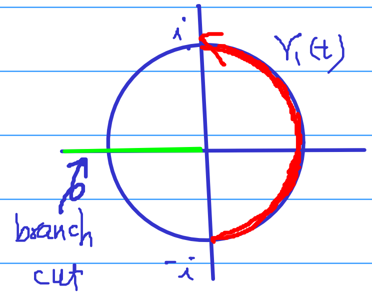
$$\therefore F(z) = \frac{z}{2} - \frac{1}{4}\sin(2z)$$

$$\Rightarrow \int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0))$$

$$= \frac{4i}{2} - \frac{1}{4}\sin(8i) - \frac{4}{2} + \frac{1}{4}\sin(8)$$

$$= \underline{2i - \tfrac{1}{4}\sin(8i) - 2 + \tfrac{1}{4}\sin(8)}$$

#2. (a) One can either use the definition of complex integral along the curve via the parametrization of $\gamma_1(t)$, or use the FTC. to compute:



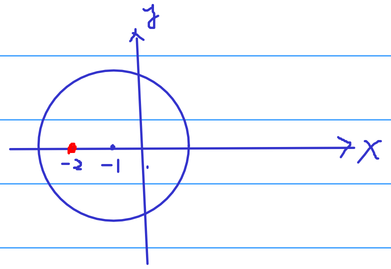
Anti-derivative z^i is $\frac{e^{(i+1)\operatorname{Log} z}}{i+1}$

$$\Rightarrow \int_{\gamma_1(t)} z^i\, dz = \left.\frac{e^{(i+1)\operatorname{Log} z}}{i+1}\right|_{z=-i}^{z=i} = \frac{i(e^{-\frac{\pi}{2}}+e^{\frac{\pi}{2}})}{i+1}$$

(b) See Lecture note #23.

#3.

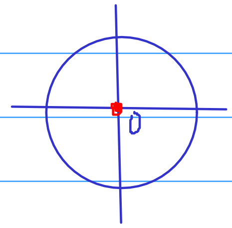
(a) $\frac{z^2}{4-z^2} = \frac{z^2/2-z}{z+2} = \frac{f(z)}{z+2}$, $f(z)=\frac{z^2}{2-z}$



Cauchy's integral formula

$$\Rightarrow \int_{C_2(-1)} \frac{z^2}{4-z^2}\, dz = 2\pi i f(-2) = 2\pi i \frac{4}{4} = 2\pi i$$

(b) $f(z) = \sin(z)$



$$\Rightarrow \int_{C_1(0)} \frac{\sin z}{z}\, dz = 2\pi i f(0) = 0$$

#4. (a) $f = u(x,y) + i v(x,y)$

Since f is holomorphic on G, u, v satisfy Cauchy-Riemann eq'n:

$$\begin{cases} \partial_x u = \partial_y v & \text{— (1)} \\ \partial_y u = -\partial_x v & \text{— (2)} \end{cases}$$

Differentiating (1) w.r.t. x and (2) w.r.t. y and since u, v are smooth functions, we have $\begin{cases} \partial_{xx} u = \partial_{xy} v \\ \partial_{yy} u = -\partial_{yx} v \end{cases}$ and $\partial_{xy} v = \partial_{yx} v$

$$\Rightarrow \quad \partial_{xx} u + \partial_{yy} u = 0$$

Similar for v:

$$\partial_{xx} v + \partial_{yy} v = 0.$$

(b) Suppose, on the contrary, that $f(z) = u + iv$ is an antider. of g, i.e.,

$$f'(z) = g = x$$

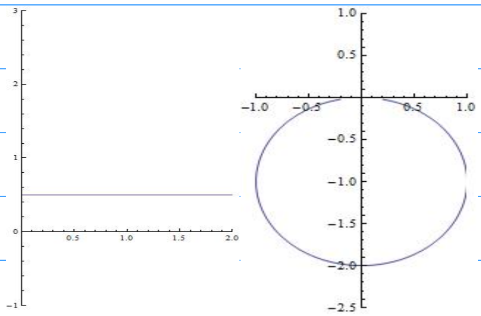
but $f'(z) = \frac{\partial f}{\partial x}(z) = \partial_x u + i \partial_x v = x \Rightarrow \partial_x u = x$

also, $f'(z) = -i \frac{\partial f}{\partial y}(z) = -i(\partial_y u + i \partial_y v) = \partial_y v - i \partial_y u = x \Rightarrow \partial_y u = 0$

$$\therefore \partial_{xx} u + \partial_{yy} u = \partial_x(x) + \partial_y(0) = 1 + 0 = 1 \neq 0$$

which is a contradiction since f has derivative $\Rightarrow$ f is holomorphic on G.

#5. (a)



(b) Let $z = x + ib$, $b > 0$

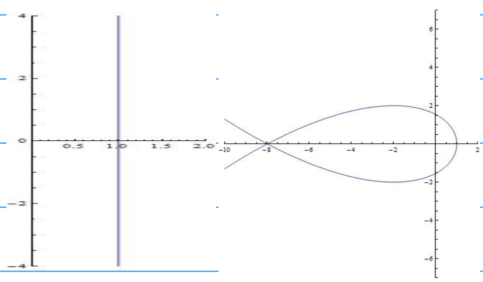
So $f(z) = \frac{1}{z} = \frac{1}{x+ib} = \frac{x}{x^2+b^2} + i \cdot \frac{-b}{x^2+b^2} := u + iv$

Note that $u^2 + v^2 = \frac{1}{x^2+b^2} = -\frac{1}{b} v$

$$\Rightarrow u^2 + v^2 + \frac{1}{b} v = 0 \Rightarrow u^2 + \left(v + \frac{1}{2b}\right)^2 = \frac{1}{4b^2} = \left(\frac{1}{2b}\right)^2 \quad (*)$$

Since $\{(u,v) : z = x+ib\}$ is the image of $y = b$ under the mapping f, we see that the image is a circle centered at $\left(0, -\frac{1}{2b}\right)$ with radius $R = \frac{1}{2b}$ by $(*)$ $\square$

#6. (a)



(b) 1° Let us assume there exist $z_1, z_2 \in \ell := \{z = 1 + iy : y \in \mathbb{R}\}$ such that

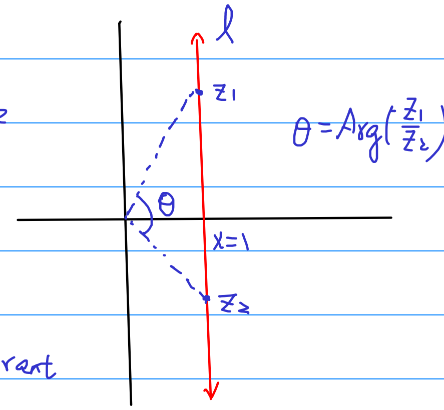
$$f(z_1) = f(z_2) \; ; \quad z_1 \neq z_2$$

This means $z_1^3 = z_2^3$ or $\left(\frac{z_1}{z_2}\right)^3 = 1 \Rightarrow \frac{z_1}{z_2} = 1^{1/3} = \left\{ e^{i\frac{2k\pi}{3}} : k = 0,1,2 \right\}$ (6.1)

Consider now the principal argument, Arg, of $\frac{z_1}{z_2}$:

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 \quad (6.2)$$

Since z_1 and z_2 are confined on the set $\ell : x = 1$, the angle between them cannot exceed π or $180°$. (see the figure below)



In view of this and the assumption that $z_1 \neq z_2$, we see that from (6.1),

$$\boxed{\operatorname{Arg} z_1 - \operatorname{Arg} z_2 = \frac{2}{3}\pi.} \quad (6.3)$$

This implies z_1 and z_2 must be in opposite quadrant (see the figure)

2° Let $z_1 = 1 + iy_1$, $z_2 = 1 - iy_2$, where $y_i > 0$, $i = 1, 2$.

For simplicity, we choose $\begin{cases} \operatorname{Arg} z_1 = \frac{\pi}{3} \\ \operatorname{Arg} z_2 = -\frac{\pi}{3} \end{cases}$

so that $\begin{cases} \underline{z_1 = 1 + i\sqrt{3}} \\ \underline{z_2 = 1 - i\sqrt{3}} \end{cases}$

※ Remark: In fact, there are infinitely many choices of such z_1 and z_2 on ℓ as long as they satisfy the condition (6.3).

(c)

$\because f'(z) = 3z^2$

$\therefore f'(z_1) = -6 + 6\sqrt{3}i$

$f'(z_2) = -6 - 6\sqrt{3}i$

(d) There are many ways to compute the angle at the intersection point, but they're intrinsically the same. Here we provide one of them. Namely, the angle of the tangent vectors

$$f'(\gamma(t)), \quad \text{where } \gamma(t) = 1 + it, \ t \in \mathbb{R}.$$

for $t = y_1$ and $t = -y_2$ and <u>notice that y_1, y_2 doesn't necessarily be the same numbers as in part (b).</u>

Hence, the tangent vectors at the intersection of the image corresponding to z_1, z_2 are:

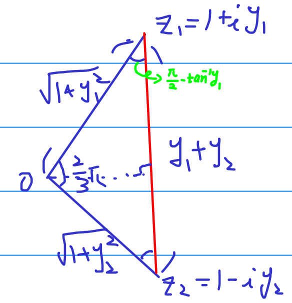
$$\vec{u} = (-6y_1, 3 - 3y_1^2) \quad \text{and} \quad \vec{v} = (6y_2, 3 - 3y_2^2)$$

Thus,

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|} = \frac{-36y_1y_2 + 9 - 9y_1^2 - 9y_2^2 + 9y_1^2y_2^2}{3\cdot(y_1^2+1)\cdot 3\cdot(y_2^2+1)}$$

$$= \frac{-4y_1y_2 + 1 - y_1^2 - y_2^2 + y_1^2y_2^2}{y_1^2y_2^2 + y_1^2 + y_2^2 + 1} \qquad (6.4)$$

Now, using "Law of sine", we have



Law of sine

$$\frac{y_1+y_2}{\sin\left(\frac{2\pi}{3}\right)} = \frac{\sqrt{1+y_2^2}}{\sin\left(\frac{\pi}{2} - \tan^{-1} y_1\right)}$$

$$\Rightarrow \frac{y_1+y_2}{\sqrt{3}/2} = \frac{\sqrt{1+y_2^2}}{1/\sqrt{1+y_1^2}} = \sqrt{1+y_1^2}\cdot\sqrt{1+y_2^2}$$

$$\Rightarrow (y_1+y_2)^2 = \frac{3}{4}(1+y_1^2)(1+y_2^2)$$

$$\Rightarrow y_1^2 + 2y_1y_2 + y_2^2 = \frac{3}{4}(1 + y_1^2 + y_2^2 + y_1^2y_2^2)$$

$$\Rightarrow 2y_1y_2 = \frac{3}{4} - \frac{1}{4}y_1^2 - \frac{1}{4}y_2^2 + \frac{3}{4}y_1^2y_2^2$$

$$\Rightarrow -4y_1y_2 = -\frac{1}{2}(1 + y_1^2 + y_2^2 + y_1^2y_2^2) \qquad (6.5)$$

Combining (6.4), (6.5) we obtain

$$\cos(\theta) = -\frac{1}{2}$$

so $\theta = \frac{2\pi}{3}$ or $120°$

$\square$