Homework 6

Due 5/23/2019

Problem 1, 2, 3 ask you to evaluate complex integrals. You can use the command **Integrate** in Mathematica to double check your results. Try this example:

```
Integrate[1/(t-1 + t*I)^2, {t, -1, 1}]
```

1. Do Problem 4.18 page 70 of the textbook.
2. Do Problem 4.19 page 70 of the textbook. **Warning:** the identity $(a^b)^c = a^{bc}$ is generally not true for complex numbers. (It is true when c is an integer.)
3. Evaluate the following integrals. The circles are positively oriented.

   (a)
   $$\int_{C_2(-1)} \frac{z^2}{4-z^2}dz$$

   (b)
   $$\int_{C_1(0)} \frac{\sin z}{z}dz$$

4. A real-valued function $u = u(x, y)$ is said to be *harmonic* if $\partial_{xx}u + \partial_{yy}u = 0$.
   (a) Let $f : G \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function. In complex standard form, $f(z) = u(z) + iv(z)$. Verify that both u and v are harmonic functions on G.
   Hint: Use Cauchy-Riemann equations.
   (b) Verify that the function $g(z) = x = \operatorname{Re} z$ has no antiderivatives on $\mathbb{C}$.
   Hint: suppose $f' = g$. Use Part (a) to draw a contradiction.

In Part (a) of Problem 5 and 6, you should use Mathematica to sketch graphs. In other parts, you can use Mathematica to assist your arguments (which is particularly useful for Part (c) of Prob. 5, and Part (b) of Prob. 6). *Make sure to write the Mathematica code you use, give explanation and some comments on the graph.* Similar treatment is done in the supplemental material "**Mapping properties of inversion function**" posted on the course website.

5. Consider the function $f(z) = \frac{1}{z}$.
   (a) Sketch the horizontal line $y = 1/2$ together with its image under f.
   (b) Verify that the image of line $y = b > 0$ is a circle. What are its center and radius?
   (c) What is the image of the half-plane $\{z : \; y > 1/2\}$ under f?
6. Consider the function $f(z) = z^3$.
   (a) Sketch the image of the vertical line ℓ: $x = 1$ under f. Note that the image path intersects itself.
   (b) Find two distinct points z_1 and z_2 on ℓ such that $f(z_1) = f(z_2)$.
   Hint: $(\frac{z_1}{z_2})^3 = 1$. Find $\operatorname{Arg} z_1$ and $\operatorname{Arg} z_2$. Then use geometry to determine z_1 and z_2.
   (c) Find $f'(z_1)$ and $f'(z_2)$.
   (d) Find the angle at which the image path intersects itself.