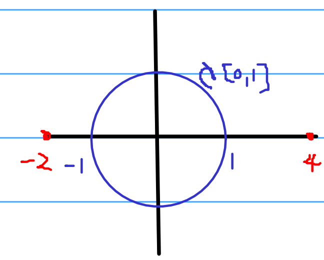
HW 7 Solutions.

#1 (Problem 4.35, p.72 of the textbook).

Compute $\displaystyle\int_{C[0,r]} \frac{dz}{z^2-2z-8}$ for $r=1$, $r=3$ and $r=5$.

Firstly, use the partial fractions expansion of the integrand:

$$\Rightarrow \int_{C[0,r]} \frac{dz}{z^2-2z-8} = \frac{1}{3}\int_{C[0,r]} \frac{1}{z-4}\,dz - \frac{1}{3}\int_{C[0,r]} \frac{1}{z+2}\,dz.$$



- Case 1: $r=1$, $\displaystyle\frac{1}{3}\int_{C[0,1]} \frac{1}{z-4}\,dz = 0$ and $\displaystyle\frac{1}{3}\int_{C[0,1]} \frac{1}{z+2}\,dz = 0$

$$\therefore \int_{C[0,1]} \frac{dz}{z^2-2z-8} = \underline{0}$$

- Case 2: $r=3$, $\displaystyle\frac{1}{3}\int_{C[0,3]} \frac{1}{z-4}\,dz = 0$

but $\displaystyle\frac{1}{3}\int_{C[0,3]} \frac{1}{z+2}\,dz = \frac{1}{3}\cdot 2\pi i = \frac{2}{3}\pi i$ since $z=-2$ is inside the region enclosed by $C[0,3]$ and Theorem 4.27 has been applied.

$$\therefore \int_{C[0,3]} \frac{dz}{z^2-2z-8} = \underline{-\frac{2}{3}\pi i}$$

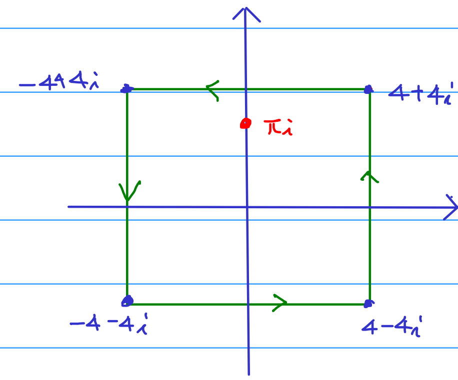
- Case 3: $r=5$. Again, using Theorem 4.27, we obtain

$$\frac{1}{3}\int_{C[0,5]} \frac{1}{z-4}\,dz = \frac{2}{3}\pi i = \frac{1}{3}\int_{C[0,5]} \frac{1}{z+2}\,dz$$

$$\Rightarrow \int_{C[0,5]} \frac{dz}{z^2-2z-8} = \underline{0}$$

#2. (Problem 5.1(b) of the textbook) Compute the integral:

$$\int_{\square} \frac{\exp(3z)}{(z-\pi i)^2}\,dz$$



where $\square$ is the boundary of the square shown on the right hand side.

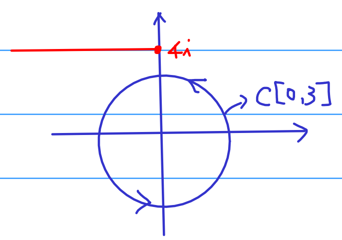
Let $f(z) = \exp(3z)$ which is holomorphic on $\overline{\square}$.

Use Cauchy's integral formula,

$$\int_{\square} \frac{f(z)}{(z-\pi i)^2}\,dz = 2\pi i \cdot f'(\pi i) = \underline{-6\pi i}$$

#3. (1) (Problem 5.3(a) of the textbook)

The branch cut of the function $\operatorname{Log}(z-4i)$ is shown below (in red)



$\Rightarrow$ Since $\operatorname{Log}(z-4i)$ is holomorphic inside the circle $C[0,3]$ and on $C[0,3]$.

Cauchy-Goursat Theorem $\Rightarrow \displaystyle\int_{C[0,3]} \operatorname{Log}(z-4i)\,dz = \underline{0}$

(2) (Problem 5.3(f))

(Method 1) By Cauchy-Goursat Theorem:

Note that $i^{z-3} = e^{(z-3)\operatorname{Log} i} = e^{\frac{\pi}{2}i\cdot(z-3)}$ is holomorphic on $\mathbb{C}$. (entire)

$\therefore$ Cauchy-Goursat $\Rightarrow \displaystyle\int_{C[0,3]} i^{z-3}\,dz = 0$

(Method 2) By definition of complex integral:

Let $\gamma(t) := 3\cos(t) + i\,3\sin(t)$, $t \in [0, 2\pi]$

Then

$$\int_{C[0,3]} i^{z-3}\,dz = \int_{C[0,3]} e^{(z-3)\operatorname{Log} i}\,dz = \int_0^{2\pi} \exp\left[-\tfrac{3}{2}\pi(\sin t) + i\tfrac{3}{2}\pi(\cos t - 1)\right]\cdot(-3\sin t + i3\cos t)\,dt$$

$$= \int_0^{2\pi} \exp\left[-\tfrac{3}{2}\pi(\sin t) + i\tfrac{3}{2}\pi(\cos t)\right]\cdot(-3\cos t - i3\sin t)\,dt$$

$$= 2\cdot\int_0^{2\pi} \frac{d}{dt}\left[e^{-\frac{3}{2}\pi\sin t + i\frac{3}{2}\pi\cos t}\right]dt$$

$$= 2\, e^{i\frac{3}{2}\pi z(t)}\Big|_{z(0)=3}^{z(2\pi)=3} = 0$$

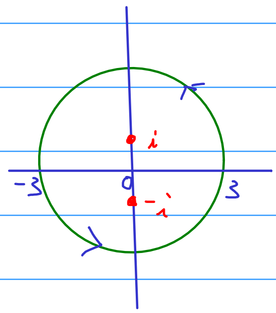
(3) (Problem 5.3 (d))

$$\int_{C[0,3]} \frac{1}{(z+4)(z^2+1)}\,dz = \frac{i}{2}\int_{C[0,3]} \frac{1}{z+4}\left(\frac{1}{z+i} - \frac{1}{z-i}\right)dz$$

$$= \frac{i}{2}\int_{C[0,3]} \frac{1/(z+4)}{z+i}\,dz - \frac{i}{2}\int_{C[0,3]} \frac{1/(z+4)}{z-i}\,dz$$

$$= \frac{i}{2}\cdot 2\pi i\,\frac{1}{4-i} - \frac{i}{2}\cdot 2\pi i\,\frac{1}{4+i}$$

$$= \underline{-\frac{2\pi}{17}\,i}$$



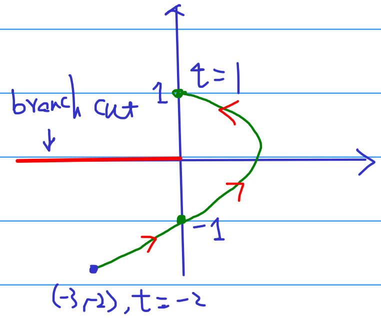
#4.

$\vec{\gamma}(t) = (1-t^2, t)$, $t \in [-2, 1]$

Let $x = 1-t^2$, $y = t$. then $x = 1-y^2$ ... a parabola

Since $\frac{1}{z^{1/2}}$ has an antiderivative in $\mathbb{C}\setminus\mathbb{R}_{\le 0}$,

namely, $\frac{d}{dz}\left(2z^{\frac{1}{2}}\right) = z^{-\frac{1}{2}}$.



Therefore,

$$\int_\gamma \frac{1}{z^{1/2}}\,dz = 2z^{1/2}\Big|_{z=-3-2i}^{z=i} = \underline{2i^{\frac{1}{2}} - 2(-3-2i)^{\frac{1}{2}}}$$

#5. (Problem 7.25 of the textbook)

(a) $$\frac{1}{1+4z} = \frac{1}{1-(-4z)} = 1+(-4z)^2+(-4z)^3+(-4z)^4+\cdots = \boxed{\sum_{k=0}^{\infty}(-4)^k z^k} := \sum a_k z^k$$

Radius of convergence:

$$|a_k|^{\frac{1}{k}} = 4 \Rightarrow \underline{R = \frac{1}{4}}$$

(b) $$\frac{1}{3-\frac{z}{2}} = \frac{1}{3}\cdot\frac{1}{1-\frac{z}{6}} = \frac{1}{3}\cdot\sum_{k=0}^{\infty}\left(\frac{z}{6}\right)^k = \boxed{\sum_{k=0}^{\infty}\frac{1}{3}\left(\frac{1}{6}\right)^k\cdot z^k} := \sum a_k z^k$$

$$|a_k|^{\frac{1}{k}} = \frac{1}{3^{1/k}}\frac{1}{6} \to \frac{1}{6} \text{ as } k\to\infty$$

$\therefore$ Radius of convergence $\underline{R=6}$

(c) First, note that $\frac{d}{dz}\left(\frac{1}{4-z}\right) = \frac{1}{(4-z)^2}$

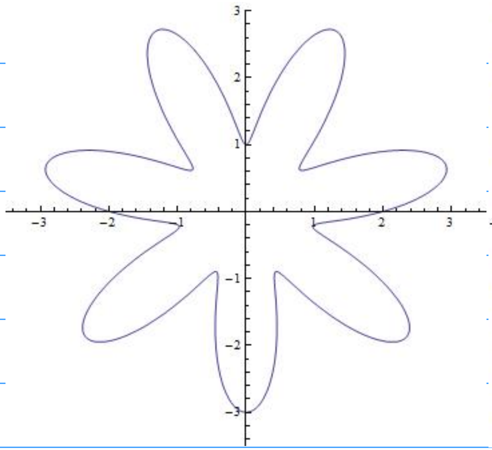
and $$\frac{1}{4-z} = \frac{1}{4}\cdot\frac{1}{1-\frac{z}{4}} = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{z}{4}\right)^k$$

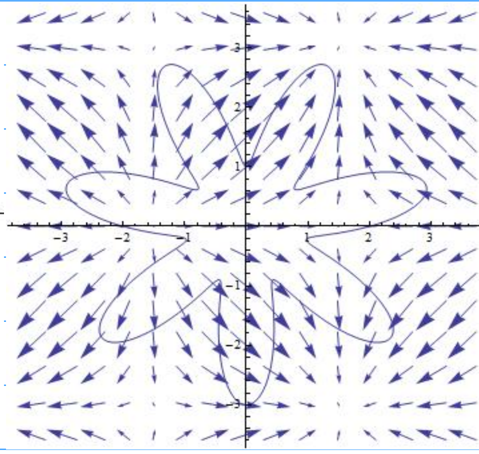
so $$\frac{1}{(4-z)^2} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{k}{4^k}z^{k-1} = \frac{1}{4}\sum_{k=0}^{\infty}\frac{k+1}{4^{k+1}}z^k$$

Therefore, power series for $\frac{z^2}{(4-z)^2}$ is $\boxed{\frac{1}{4}\sum_{k=2}^{\infty}\frac{k-1}{4^{k-1}}z^k}$

Radius of convergence $R = \frac{1}{\lim_{k\to\infty}\left|\frac{k-1}{4^{k-1}}\right|^{\frac{1}{k}}} = \frac{1}{1/4} = \underline{4.}$

#6. (a) & (b)





(c)

Pólya vector field of f = $\bar{f}(z) = \cos(x) - i\sin(y)$

or $(\cos(x), -\sin(y))$

(d) & (e)

Notice that $\int_\gamma \bar{f}(z)\,dz = W[f,\gamma] + i\,F[f,\gamma]$

$:=$ (work done by f along γ) $+ i\cdot$(flux of f through γ)

- $W[f,\gamma] = \int_0^{2\pi} (u(x(t),y(t)), v(x(t),y(t)))\cdot(x', y')\,dt$, where we denote f by $f = u + iv$

$$= \int_0^{2\pi} x'(t)\cdot\cos(x(t)) + y'(t)\cdot\sin(y(t))\,dt$$

$$= \int_0^{2\pi} \frac{d}{dt}\left[\cos(x(t)) + \sin(y(t))\right] dt = 0$$

In other words, the complex integral of the Pólya field: $\int_\gamma \bar{f}(z)\,dz$ is equal to i times the flux of f across γ:

$$\int_\gamma \bar{f}(z)\,dz = i\,F[f,\gamma]$$