HW 8 Solutions.

#1.

(a) Use ratio test on this one :

$$\lim_{n\to\infty}\left|\frac{2^{n+1}z^{n+1}}{1+3^{n+1}}\cdot\frac{1+3^n}{2^n\cdot z^n}\right| = \lim_{n\to\infty} 2\cdot\frac{\frac{1}{3^n}+1}{\frac{1}{3^n}+3}\cdot|z| = \frac{2}{3}|z| < 1$$

$$\Rightarrow |z| < \frac{3}{2}$$

$\therefore$ The radius of convergence $R=\frac{3}{2}$

The region of convergence : $D_{3/2}(0)$

(b) Using root test :

$$\lim_{n\to\infty}\left|\frac{z^n}{(\sqrt{3}+i)^n}\right|^{\frac{1}{n}} = \frac{|z|}{2} < 1 \Rightarrow |z| < 2$$

$\therefore$ The radius of convergence : $R=2$ $\qquad |(\sqrt{3}+i)^n| = |\sqrt{3}+i|^n$

The region of convergence : $D_2(0)$

#2. Rewrite $f(z) = \dfrac{3}{-z^2+z+2}$ as the following :

$$f(z) = \frac{3}{(1+z)(2-z)} = \frac{1}{3}\cdot 3\cdot\left(\frac{1}{1+z}+\frac{1}{2-z}\right) = \frac{1}{1-(-z)}+\frac{1}{2-z}$$

- $\dfrac{1}{1-(-z)} = 1+(-z)+(-z)^2+(-z)^3+\cdots = \displaystyle\sum_{k=0}^{\infty}(-1)^k\cdot z^k, \quad |z|<1$
- $\dfrac{1}{2-z} = \dfrac{1}{2}\dfrac{1}{1-z/2} = \dfrac{1}{2}\left[1+\left(\dfrac{z}{2}\right)+\left(\dfrac{z}{2}\right)^2+\cdots\right] = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty}\left(\dfrac{z}{2}\right)^k = \displaystyle\sum_{k=0}^{\infty}\dfrac{z^k}{2^{k+1}}, \quad |z|<2$

Therefore, the Laurent series representation of $f(z)$ about 0 fall into 3 categories:

(1.) $f(z) = \displaystyle\sum_{k=0}^{\infty}\left[(-1)^k+\frac{1}{2^{k+1}}\right]z^k, \quad |z|<1$

(2.) $f(z) = \dfrac{1}{2-z}+\dfrac{1}{z}\dfrac{1}{1+\frac{1}{z}} = \left(\displaystyle\sum_{k=0}^{\infty}\dfrac{z^k}{2^{k+1}}\right)+\dfrac{1}{z}\displaystyle\sum_{k=0}^{\infty}\left(\dfrac{-1}{z}\right)^k \qquad$ (since $|\frac{1}{z}|<1$)

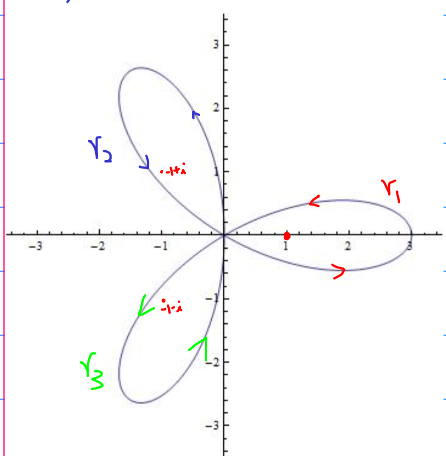
$$= \sum_{k=-\infty}^{-1}(-1)^{k+1}z^k + \sum_{k=0}^{\infty}\frac{1}{2^{k+1}}\cdot z^k, \quad \text{if } 1<|z|<2$$

(3) $f(z) = \frac{-1}{z}\cdot\frac{1}{1-\frac{2}{z}} + \frac{1}{z}\,\frac{1}{1+\frac{1}{z}} = \frac{-1}{z}\cdot\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k + \sum_{k=-\infty}^{-1}(-1)^{k+1}\cdot z^k$ (since $\left|\frac{2}{z}\right|<1$)

$$= -\sum_{k=0}^{\infty} 2^k\, z^{-(k+1)} + \sum_{k=-\infty}^{-1}(-1)^{k+1}\cdot z^k$$

$$= -\sum_{k=-\infty}^{-1}\frac{1}{2^{k+1}}z^k + \sum_{k=-\infty}^{-1}(-1)^{k+1}z^k = \sum_{k=-\infty}^{-1}\left[(-1)^{k+1}-\frac{1}{2^{k+1}}\right]\cdot z^k, \text{ if } |z|>2$$

#3. (a)



(b) $$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz$$

Note that

$$f(z) = \frac{1}{(z-1)(z-(-1+i))(z-(-1-i))}$$

$$\Rightarrow \int_\gamma f(z)\,dz = \int_{\gamma_1}\frac{f_1(z)}{z-1}\,dz + \int_{\gamma_2}\frac{f_2(z)}{z-(-1+i)}\,dz + \int_{\gamma_3}\frac{f_3(z)}{z-(-1-i)}$$

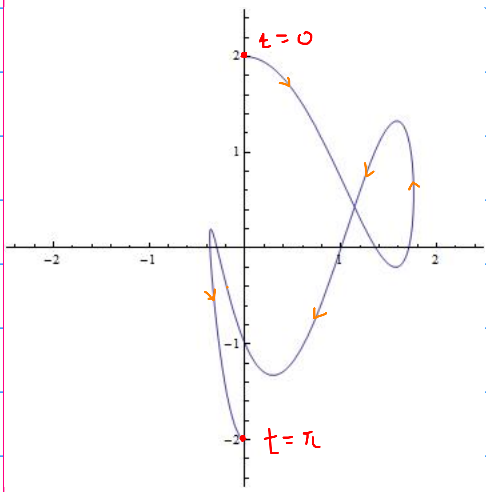
where $f_1(z) = \frac{1}{(z-(-1+i))(z-(-1-i))}$; $f_2(z) = \frac{1}{(z-1)(z-(-1-i))}$; $f_3(z) = \frac{1}{(z-1)(z-(-1+i))}$

$$\therefore \int_\gamma f(z)\,dz = 2\pi i\left\{f_1(1) + f_2(-1+i) + f_3(-1-i)\right\}$$

$$= 2\pi i\cdot\left\{\frac{1}{5} + \frac{-1+2i}{10} + \frac{-1-2i}{10}\right\} = \underline{0}$$

#4.

(a)



(b)

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\therefore f(z) = \frac{e^z}{z^3} = \sum_{n=-3}^{\infty} \frac{z^n}{(n+3)!}$$