Lecture 1 (4/1/2019)

A brief comparison:

Calculus of functions of real variables

Object:
- curves
- $f: D \subset \mathbb{R} \to \mathbb{R}$

Topics:
- slope
- monotonicity (increasing, decreasing)
- concavity / convexity
- extrema
- area under curve
- length of curve
- solid / surface of revolution

most properties of a function can be interpreted on it graph

Calculus of functions of vector variables

Object:
- surfaces
- $f: D \subset \mathbb{R}^n \to \mathbb{R}^m$

Topics:
- directional slope (partial derivatives)
- normal vectors (gradient)
- area / volume (multiple integrals)
- flux across surface

similar

Calculus of functions of complex variables

Object:
- conformal transformation
- $f: D \subset \mathbb{C} \to \mathbb{C}$

Topics:
- calculus (of diff. functions): limit, continuity, differentiability, integrability.
- unifies several elementary functions: exp, log, sin, cos, ...
- conformal transformation

Properties of a diff. function (called holomorphic function) are not easily interpretted by its "graph" (transformation on the plane).

A function $f: D \subset \mathbb{C} \to \mathbb{C}$ can be visualized in various ways. Properties of (complex) differentiable functions are so rich that most of them are difficult to intepret through its corresponding transformation.

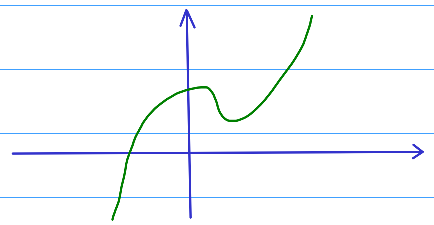
- Conformal maps has applications in:
  - harmonic functions: potential flow, heat transfer, ...
  - Riemann surfaces,
  - geometry, ...

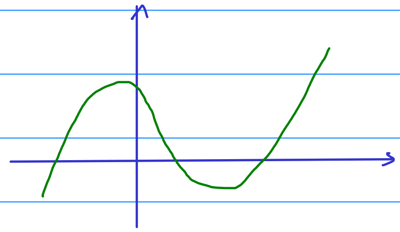
- Calculus of complex functions has applications in:
  - Fourier analysis,
  - Schroedinger equation, quantum physics,
  - number theory, ...

* Introduction to complex numbers:

Effort to find roots of a cubic polynomial by S. Ferro, N. Tatarglia and G. Carnado in 1500s:

$x^3 + 3x + 4 = 0$ $\qquad\qquad$ $x^3 - 3x + 1 = 0$





How to find roots to $x^3 + ax + b = 0$?

Write $x = u + v$.

$$\underbrace{(u+v)^3}_{} + a(u+v) + b = 0$$

$$= u^3 + 3u^2 v + 3uv^2 + v^3$$

$$= u^3 + v^3 + 3uv(u+v)$$

$$\rightsquigarrow \quad u^3 + v^3 + (3uv + a)(u+v) + b = 0 \qquad (1 \text{ eq}, 2 \text{ unknowns})$$

Suppose $3uv + a = 0$.

Then $u^3 + v^3 + b = 0$

Thus,
$$\begin{cases} u^3 + v^3 = -b \\ u^3 v^3 = -\dfrac{a^3}{27} \end{cases}$$

u^3 and v^3 are roots of $t^2 + bt - \dfrac{a^3}{27} = 0$

$$\Delta = b^2 + \frac{4}{27} a^3$$

For $x^3 + 3x + 4 = 0$: $a = 3$, $b = 4$

$$\Delta = 20$$

$$u^3, v^3 = \frac{-4 \pm \sqrt{20}}{2} = -2 \pm \sqrt{5}$$

$$u, v = \sqrt[3]{-2 \pm \sqrt{5}}$$

$$x = u + v = \sqrt[3]{-2+\sqrt{5}} + \sqrt[3]{-2-\sqrt{5}}$$

For $x^3 - 3x + 1 = 0$: $a = -3$, $b = 1$

$$\Delta = -3 < 0$$

$$u^3, v^3 = \text{“}\frac{-1 \pm \sqrt{-3}}{2}\text{”}$$