Lecture 10 (4/22/2019)

* Inverse of sine :

$$\arcsin z = w \rightsquigarrow \sin w = z \rightsquigarrow \frac{e^{iw} - e^{-iw}}{2i} = z$$

Put $u = e^{iw}$. Then $\dfrac{u - u^{-1}}{2i} = z \rightsquigarrow u^2 - 2uiz + 1 = 0$

$$\rightsquigarrow u = iz + \sqrt{1-z^2}$$

$$\rightsquigarrow iw = \log u = \log\left(iz + \sqrt{1-z^2}\right)$$

$$\rightsquigarrow \arcsin z = \frac{1}{i}\log\left(iz + \sqrt{1-z^2}\right)$$

Multivalues come from the log and "$\sqrt{\ }$" functions.

Similarly,

$$\arccos z = \frac{1}{i}\log\left(z + i(1-z^2)^{1/2}\right)$$

$$\arctan z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$$

Ex: compute $\arcsin i$

$$\arcsin i = \frac{1}{i}\log\left(i\,i + \sqrt{1-i^2}\right) = \frac{1}{i}\log\Big(-1 + \underbrace{\sqrt{2}}_{\text{complex root}}\Big)$$

$$= \frac{1}{i}\log\Big(-1 \pm \underbrace{\sqrt{2}}_{\text{real root}}\Big)$$

If plus sign is selected,

$$\arcsin i = \frac{1}{i}\log\Big(\underbrace{-1+\sqrt{2}}_{\text{positive}}\Big) = \frac{1}{i}\left(\ln(-1+\sqrt{2}) + i\,k2\pi\right)$$

$$= k2\pi - i\ln(-1+\sqrt{2})$$

If minus sign is selected,

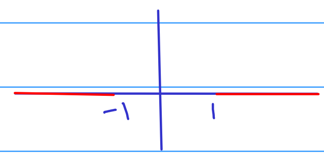
$$\arcsin i = \frac{1}{i}\log\Big(\underbrace{-1-\sqrt{2}}_{\text{negative}}\Big) = \frac{1}{i}\left(\ln(1+\sqrt{2}) + i\pi + i\,k2\pi\right)$$

$$= k2\pi + \pi - i\ln(1+\sqrt{2})$$

Conclusion: $\arcsin i = \{ k2\pi - i\ln(-1+\sqrt{2}),\ k2\pi + \pi - i\ln(1+\sqrt{2}) \ : \ k \in \mathbb{Z} \}$
$$= \{ n\pi - i(-1)^n \ln(-1+\sqrt{2}) \ : n \in \mathbb{Z} \}$$

* How to define a single branch for the $\arcsin z$?

$$\operatorname{Arcsin} z = \frac{1}{i} \operatorname{Log}\Big(iz + \underbrace{\sqrt{1-z^2}}_{\text{principal branch}} \Big)$$

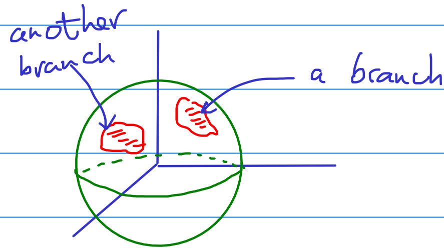


Up to this point, we have considered some basic functions on $\mathbb{R}$ and their extensions to $\mathbb{C}$.

exp, sin, cos, tan, power (z^n, $n \in \mathbb{Z}$) : single-valued functions

log, arcsin, arccos, arctan, z^a ($a \in \mathbb{C}$) : multi-value functions

Branch cuts are introduced to create a single-valued function (branch) out of a multi-valued function (likened to a tree). There are many way to define a branch. Think of a sphere: the sphere itself is not a graph of a function. But a hemisphere is. It is also a maximal branch in sense that one can't extend the hemisphere and still have a graph of a function.



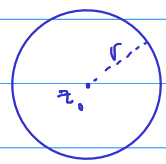
The functions have much richer geometric properties than their real counterparts (as already seen in previous lectures and homework). For example, these functions are conformal transformations (i.e. angle-preserving).

To explain these geometric properties analytically, we need more tools, for example, continuity, derivatives, integrals,... These concepts are defined on functions $f : D \subset \mathbb{C} \to \mathbb{C}$.

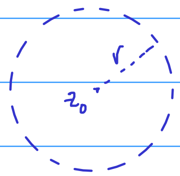
A subset $D \subset \mathbb{C}$ has richer geometric properties than a subset $I \subset \mathbb{R}$. These properties of D influence all functions defined on it. We introduce some geometric/topological properties as follows:

- Circle centered at z_0 with radius r: $C_r(z_0)$
  Note that the textbook use notation $C[z_0, r]$.

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$$



- Open disk centered at z_0 with radius r: $D_r(z_0)$
  The textbook uses notation $D[z_0, r]$.



$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

- Closed disk: $\overline{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$
- Let $G \subset \mathbb{C}$.
  * A point $a \in G$ is said to be interior point of G if $D_r(a) \subset G$ for some $r > 0$.