

Lecture 11 (4/24/2019)

Calculus of real-valued function $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ involves the notion of

limit is the central notion

- limit $\lim_{x \rightarrow a} f(x) = L$
- continuity $\lim_{x \rightarrow x_0} f(x) = f(x_0)$
- derivative $f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$
- integral $\int f(x) dx$; $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \frac{b-a}{n}$

We will consider each of these notions for complex-valued functions $f: G \subset \mathbb{C} \rightarrow \mathbb{C}$. Let us first introduce some topological terminology:

* A point $a \in G$ is said to be **interior point** of G if $D_r(a) \subset G$ for some $r > 0$.

The set of interior points of G is denoted by G° , called the **interior** of G .

• A point $d \in \mathbb{C}$ is said to be **boundary point** of G if $\begin{cases} D_r(d) \cap G \neq \emptyset \\ D_r(d) \cap G^c \neq \emptyset \end{cases} \quad \forall r > 0$

Here $G^c = \mathbb{C} \setminus G$ is the complement of G in \mathbb{C} .

• The set of boundary points of G is denoted by ∂G .

Observation: $G^\circ \subset G \subset \underbrace{G^\circ \cup \partial G}_{\bar{G}}$

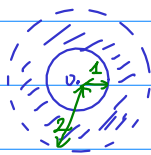
\bar{G} , called **closure** of G

• A set $G \subset \mathbb{C}$ is said to be **open** if every point of it is an interior point. (or equivalently, if it contains no boundary points)

In other words, $G^\circ = G$.

• A set $G \subset \mathbb{C}$ is said to be **closed** if it contains every boundary point.

Ex: $G = \{z: 1 \leq |z| < 2\}$

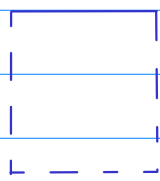


Interior points: $G^\circ = \{z: 1 < |z| < 2\}$

Boundary points: $\partial G = \{z: |z|=1\} \cup \{z: |z|=2\}$
 $= C_1(0) \cup C_2(0)$

Closure of G : $\bar{G} = G \cup \partial G = \{z: 1 \leq |z| \leq 2\}$

Ex: $G = \{z = x + iy: -1 < x < 1, 0 < y \leq 3\}$



$\partial G = \{z: x = -1, 0 \leq y \leq 3\}$

$\cup \{z: x = 1, 0 \leq y \leq 3\}$

$\cup \{z: -1 \leq x \leq 1, y = 0\}$

$\cup \{z: -1 \leq x \leq 1, y = 3\}$

$G^\circ = \{z: -1 < x < 1, 0 < y < 3\}$

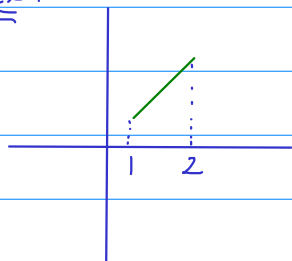
Tips: If a set $G \subset \mathbb{C}$ is described by inequalities (as the above example), it is usually the case that the points corresponding to strict inequalities are interior points, the points with at least one equality are boundary points.

Caution: this is not always the case, for example

$$G = \{z: 0 \leq |z| < 1\} = D_1(0)$$

is open, although " $0 \leq |z|$ " is not a strict inequality.

Ex:



$$G = \{z = x + iy: x = y, 1 \leq x \leq 2\}$$

$$G^\circ = \emptyset$$

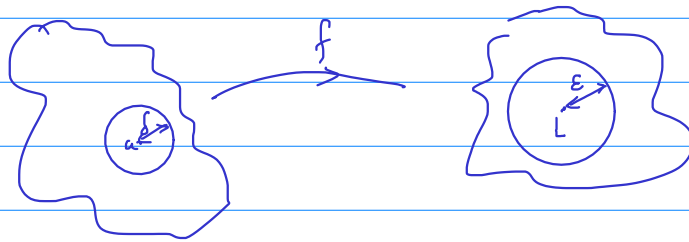
$$\partial G = G$$

* Limit:

$\lim_{x \rightarrow a} f(x) = L \rightsquigarrow$ behavior of f near a , but not at a .

Formal definition:

$\lim_{x \rightarrow a} f(x) = L$ (the limit of f as x approaches a is equal to L)
if: for each $\varepsilon > 0$ there exists $\delta > 0$ (depending on ε) such that
if $x \in D_\delta(a)$ then $f(x) \in D_\varepsilon(L)$.

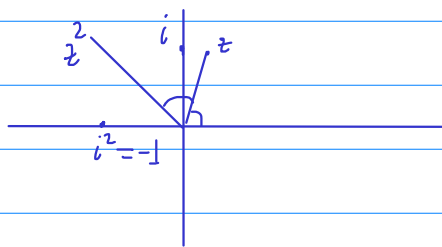


This definition applies also for functions with real variable. The "disk" in one-dimension is interval:



$\lim_{x \rightarrow a} f(x) = L$ means: for any prescribed error ε , one can find a vicinity of radius δ around a such that f shoots every point in this vicinity to a point within distance ε from L .

Ex: $f(z) = z^2$
Find $\lim_{z \rightarrow i} f(z)$.



Guess: $\lim_{z \rightarrow i} f(z) = i^2 = -1$

Estimate $|f(z) - (-1)|$ (the distance between $f(z)$ and $L = -1$):

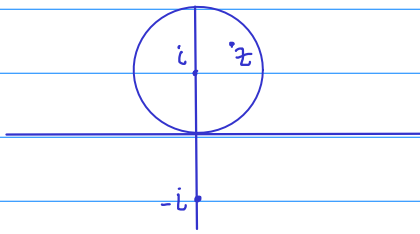
$$|f(z) + 1| = |z^2 + 1| = |(z-i)(z+i)| = |z-i||z+i|$$

Let $\varepsilon > 0$.

want to find $\delta > 0$ such that if $|z-i| < \delta$ (z is with vicinity of radius δ of i)

then $|z-i||z+i| < \varepsilon$.

Note: $|z-i||z+i| \leq \delta|z+i|$



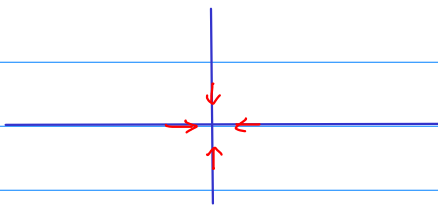
$$\begin{aligned} |z+i| &= |z-i+2i| \\ &\leq |z-i| + |2i| \\ &< \delta + 2 \\ &< 3 \quad (\text{suppose } \delta < 1). \end{aligned}$$

Then $|f(z) - (-1)| < 3\delta$ provided that $|z-i| < \delta$ and $\delta < 1$.

Pick any ε such that $0 < \varepsilon < \min\{1, \varepsilon/3\}$. Therefore, $\lim_{z \rightarrow i} f(z) = -1$.

Ex 2: $f(z) = \frac{\bar{z}}{|z|}$

Find $\lim_{z \rightarrow 0} f(z)$.



If $z \rightarrow 0$ on the positive real axis then

$$f(z) = \frac{\bar{z}}{|z|} = \frac{z}{z} = 1 \rightarrow 1 \text{ as } z \rightarrow 0.$$

If $z \rightarrow 0$ on the negative real axis then

$$f(z) = \frac{\bar{z}}{|z|} = \frac{z}{-z} = -1 \rightarrow -1 \text{ as } z \rightarrow 0$$

If $z \rightarrow 0$ on the positive imaginary axis then

$$f(z) = f(ia) = \frac{\overline{ia}}{|ia|} = \frac{-ia}{a} = -i \rightarrow -i \text{ as } z \rightarrow 0$$

If $z \rightarrow 0$ on the negative imaginary axis then

$$f(z) = f(-ia) = \frac{\overline{-ia}}{|-ia|} = \frac{ia}{a} = i \rightarrow i \text{ as } z \rightarrow 0$$

We see that f has no limit as $z \rightarrow 0$.