Lecture 12 (4/26/2019)

A function $f = f(z)$ can be viewed as a pair $(u(x,y), v(x,y))$

where $u(x,y) = \operatorname{Re} f(x,y)$

$v(x,y) = \operatorname{Im} f(x,y)$

• Observation: $\lim_{z \to a} f(z) = L \equiv (L_1, L_2)$ if and only if

$$\begin{cases} \lim_{(x,y)\to(x_0,y_0)} u(x,y) = L_1 \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) = L_2 \end{cases}$$

here $a \equiv (x_0, y_0)$

In other words, the real part approaches the real part; the imaginary part approaches the imaginary part.

* Continuity:

$f : G \subset \mathbb{C} \to \mathbb{C}$ is said to be continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

f is said to be continuous on G if it is continuous at every point of G.

Thm:

Write $f(z) = u(x,y) + i\,v(x,y)$. Then f is cont. at $z_0 = (x_0, y_0)$ if and only if both u and v are continuous at (x_0, y_0).

Ex:

$$f(z) = z^2 \rightsquigarrow \begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{cases}$$

Because u and v are continuous, f is continuous.

Similarly, the function $g(z) = z^n$ $(n \in \mathbb{N})$ is continuous:

$$g(z) = (x+iy)^n = \underbrace{(\text{- - - - -})} + i\,\underbrace{(\text{- - - - -})}$$

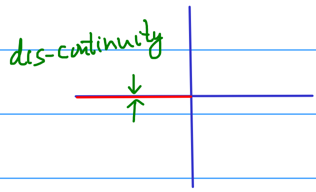
polynomials in x and y

Ex: $f(z) = e^z = e^x e^{iy} = \underbrace{e^x \cos y}_{\text{cont.}} + i \underbrace{e^x \sin y}_{\text{cont.}}$

$$\cos z = \frac{e^z + e^{-z}}{2} = u(x,y) + i v(x,y)$$

continuous

Ex: $f(z) = \operatorname{Arg}(z)$



$z = -2$

$\operatorname{Arg}(-2 + i\varepsilon) \approx \pi$

$\operatorname{Arg}(-2, -i\varepsilon) \approx -\pi$

} discontinuous at -2

Arg is undefined at $z = 0$.

For $z = x + iy$:

- if z is in right half plane,

$$\operatorname{Arg} z = \underbrace{\arctan\left(\frac{y}{x}\right)}_{u(x,y) + i.0} \rightsquigarrow \text{Continuous}$$

- if z is in the upper half plane,

$$\operatorname{Arg} z = \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right) \rightsquigarrow \text{cont.}$$

- if z is in the lower half plane,

$$\operatorname{Arg} z = -\arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right) \rightsquigarrow \text{cont.}$$

Ex:

$$f(z) = \frac{\bar{z}}{|z|} \rightsquigarrow \begin{cases} u(x,y) = \dfrac{x}{\sqrt{x^2 + y^2}} \\ v(x,y) = \dfrac{-y}{\sqrt{x^2 + y^2}} \end{cases}$$

u is not continuous at $(0,0)$. Thus, f is not cont. at 0.

* Properties of limits:

Limit of complex functions satisfies the same algebraic properties as limit of real functions.

- Additive:

$$\lim_{z\to a} (f(z) + g(z)) = \lim_{z\to a} f(z) + \lim_{z\to a} g(z)$$

- Scalar-multiplicative:

$$\lim_{z\to a} c f(z) = c \lim_{z\to a} f(z)$$

- Mutiplicative:

$$\lim_{z\to a} f(z)\, g(z) = \lim_{z\to a} f(z) \lim_{z\to a} g(z)$$

- Inverse: $$\lim_{z\to a} \frac{1}{f(z)} = \frac{1}{\lim_{z\to a} f(z)}$$

* Parallel properties of continuity:

$f, g : G \subset \mathbb{C} \to \mathbb{C}$ continuous at $x_0 \in G$

- $f+g$, cf, fg, $\frac{f}{g}$ are continuous at x_0.

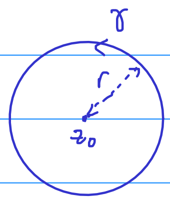
- $f(g(z))$ cont. at a if g is cont. at a and f is cont. at $f(a)$.

Consequences:

- Rational functions $\frac{P(z)}{Q(z)}$ are cont.

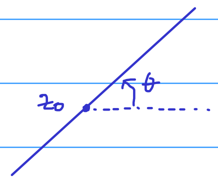
- $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ is cont. on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.
- $z^i = e^{i \operatorname{Log} z}$ is cont. on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Consider functions $\gamma : I \subset \mathbb{R} \to \mathbb{C}$. These are curves (paths) on the complex plane.



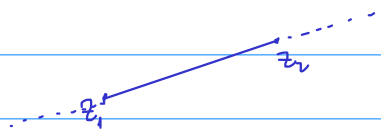
Circle centered at z_0 with radius r:

$$\gamma(t) = z_0 + re^{it}, \quad 0 \leq t \leq 2\pi$$



Line passing through z_0 with slope θ:

$$\gamma(t) = z_0 + t\,e^{i\theta}, \quad t \in \mathbb{R}$$



Segment from z_1 to z_2:

$$\gamma(t) = z_1 + t(z_2 - z_1), \quad 0 \leq t \leq 1.$$

Line from z_1 to z_2: same formula, but $t \in \mathbb{R}$.