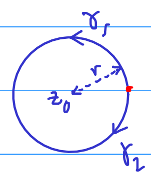
Lecture 13 (4/29/2019)

A function $\gamma : I \subset \mathbb{R} \to \mathbb{R}$ is said to be a *path* if it is continuous.



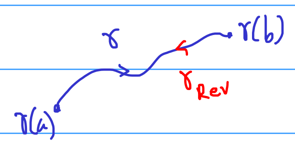
$\gamma_1(t) = z_0 + re^{it} \qquad 0 \le t \le 2\pi \qquad$ (Counter-clockwise)

$\gamma_2(t) = z_0 + re^{-it} \qquad 0 \le t \le 2\pi \qquad$ (clockwise)

How to go travel more quickly (from example going back to the original point after time π) ?

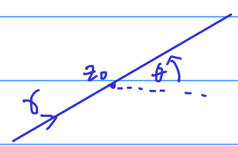
Scale time:

$$\gamma(t) = z_0 + re^{i2t} \quad (0 \le t \le \pi)$$

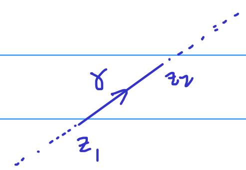


How to reverse the direction of γ ?

$$\gamma_{rev}(t) = \gamma(a+b-t) \qquad (a \le t \le b)$$



$$\gamma(t) = z_0 + te^{i\theta} \qquad (t \in \mathbb{R})$$

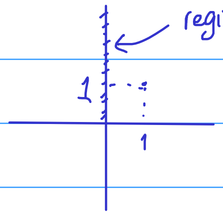


$$\gamma(t) = z_1 + t(z_2 - z_1), \qquad 0 \le t \le 1$$

Ex:

Find the region of continuity of $f(z) = \left(\frac{z}{z+1}\right)^z$

where the branch $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right]$ of the argument is used.



region of discontinuity of the branch $\overline{\arg}$

$$\overline{\arg}(1+i) = \frac{\pi}{4} + 2\pi = \frac{9\pi}{4} \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right].$$

By definition,

$$f(z) = \exp\left(z \,\overline{\log}\, \frac{z}{z+1}\right)$$

where $\mathrm{Log}\,\frac{z}{z+1} = \ln\left|\frac{z}{z+1}\right| + i\arg\left(\frac{z}{z+1}\right)$.

The rational function $\frac{z}{z+1}$ is continuous everywhere except at -1.

In addition, need to exclude of z's such that $\frac{z}{z+1} \in i\mathbb{R}_{\geq 0}$.

Write: $\frac{z}{z+1} = it$ where $t \geq 0$

$\rightsquigarrow\ z = itz + it$

$\rightsquigarrow\ z = \frac{it}{1-it}$

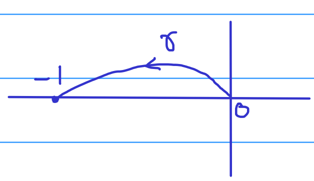
Region of continuity of f is $\mathbb{C}$ minus the set

$$\{-1\} \cup \left\{ \frac{it}{1-it} ;\ t \geq 0 \right\}$$

How to visualize this set?

Write in standard form:

$$\frac{it}{1-it} = \frac{it(1+it)}{1+t^2} = \underbrace{-\frac{t^2}{1+t^2} + i\,\frac{t}{1+t^2}}_{\text{this is a curve}}$$



On Mathematica:

ParametricPlot[{ $\frac{-t^2}{1+t^2}$, $\frac{t}{1+t^2}$ }, {t, 0, 10}]

In fact, this is the upper half of the ellipse

$$\left(x+\tfrac{1}{2}\right)^2 + 4y^2 = 1.$$

* Velocity of a path

$\gamma : [a,b] \to \mathbb{C}$

How to define $\gamma'(t)$?

Write $\gamma(t) = x(t) + i\,y(t) \equiv (x(t), y(t))$.

$$\gamma'(t_0) := \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{x(t) - x(t_0)}{t - t_0} + i \lim_{t \to t_0} \frac{y(t) - y(t_0)}{t - t_0}$$

$$= x'(t_0) + i y'(t_0)$$

$$\rightsquigarrow \quad \gamma'(t) = (x'(t), y'(t))$$

Derivative of a function $\gamma : I \subset \mathbb{R} \to \mathbb{C}$ is obtained by taking derivative of each component.

Note: Rolle's theorem doesn't hold for complex-valued functions: $\gamma(a) = \gamma(b)$ doesn't necessarily imply that there exists $c \in (a,b)$ such that $\gamma'(c) = 0$.

Ex: the circle $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$

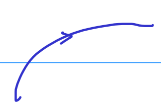
$\gamma(0) = \gamma(2\pi) = 1 + 0i$

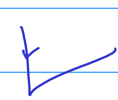
but $\gamma'(t) = (\cos t)' + i(\sin t)' = -\sin t + i\cos t \neq 0$ for all t.

A path $\gamma : [a,b] \to \mathbb{C}$ is said to be smooth if $\gamma'(t)$ exists and is continuous on $[a,b]$, with the convention that

$$\gamma'(a) = \lim_{t \to a^+} \frac{\gamma(t) - \gamma(a)}{t - a}$$

$$\gamma'(b) = \lim_{t \to b^-} \frac{\gamma(t) - \gamma(b)}{t - b}$$

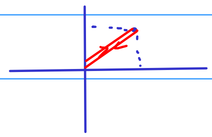

smooth


non-smooth

A path $\gamma : [a,b]$ is said to be regular if it is smooth and $\gamma'(t) \neq 0$ for all $t \in [a,b]$.

In other words, a regular path never slows down to a stop.

$\gamma(t) = (t^2, t^2)$, $-1 \leq t \leq 1$



not regular since $\gamma'(0) = 0$.

This path slows down to a stop at $t=0$, then it reverses itself.