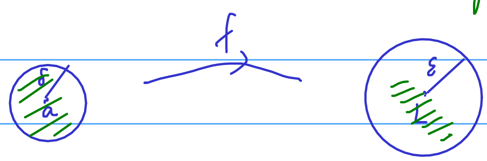
Lecture 14 (5/1/2019)

Recall the definition of limits:

$\lim_{z\to a} f(z) = L$ means:

For each $\varepsilon > 0$, there exists $\delta > 0$ depending on ε such that $|z-a|<\delta$ implies $|f(z)-L|<\varepsilon$.



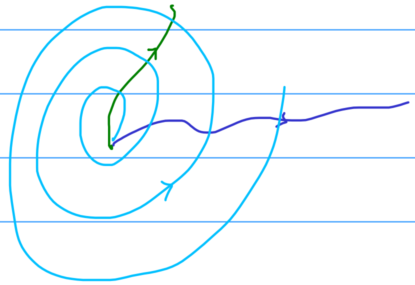
* Limit at infinity:

$\mathbb{R}$ $-\infty$ ∞

On $\mathbb{R}$, there are two "types" of infinity ($-\infty$ and ∞). This is due to the order of real numbers. There is no order for complex numbers. The only type of infinity we consider is a "point" infinitely far from the origin:

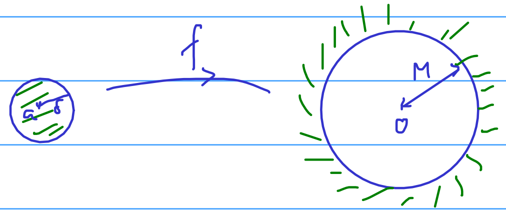
$$z \to \infty \quad \text{if} \quad |z| \to \infty.$$

Note that there are many ways for z to approach infinity.



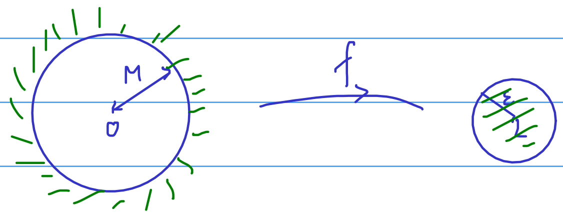
• $\lim_{z\to a} f(z) = \infty$ means:

For each $M > 0$, there exists $\delta > 0$ depending on M such that $|z-a|<\delta$ implies $\underbrace{|f(z)| > M}_{\text{"}f(z)\text{ being close to }\infty\text{"}}$.



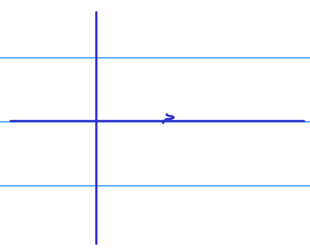
- $\lim_{z\to\infty} f(z) = L$ means:

For each $\varepsilon > 0$, there exists $M > 0$ depending on ε such that $|z| > M$ implies $|f(z) - L| < \varepsilon$.



Ex: Find $\lim_{z\to\infty} \frac{1}{z}$.

- Method 1: use definition



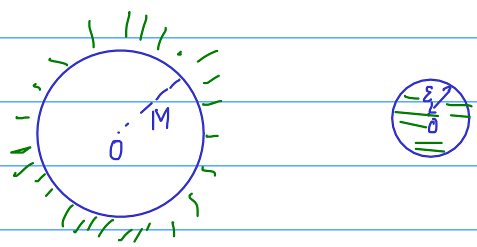
First we need to guess what the limit is.

Let z go to infinity on the real axis. In other words, let z travel on the path $\gamma : z(t) = t \in \mathbb{R}$.

$$\lim_{t\to\infty} \frac{1}{t} = 0$$

Thus, $\lim_{z\to\infty} \frac{1}{z}$ if exists must be zero. Now we use definition to show that

$$\lim_{z\to\infty} \frac{1}{z} = 0.$$



Let $\varepsilon > 0$ (the prescribed error). Find $M > 0$ such that "$|z| > M$ implies $\left|\frac{1}{z} - 0\right| < \varepsilon$".

We see that $\left|\frac{1}{z} - 0\right| = \frac{1}{|z|} < \varepsilon$ provided that $|z| > \frac{1}{\varepsilon}$.

Pick $M = 1/\varepsilon$.

Method 2: Consider limit of each component (real & imaginary part)

$$\frac{1}{z} = \frac{1}{x+iy} = \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} + i\,\underbrace{\frac{-y}{x^2+y^2}}_{v(x,y)}$$

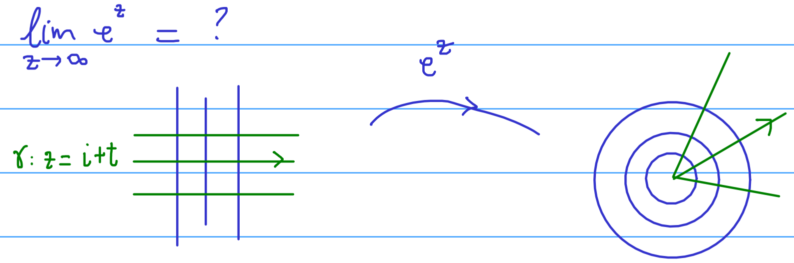
$z \to \infty$ is interpreted as $|z| \to \infty$, or $\sqrt{x^2+y^2} \to \infty$.

Need to find limit of $u(x,y)$ and $v(x,y)$ as $\underbrace{(x,y) \to \infty}_{\sqrt{x^2+y^2} \to \infty}$.

$$0 \le \frac{|x|}{x^2+y^2} = \underbrace{\frac{|x|}{\sqrt{x^2+y^2}}}_{\le 1} \ \underbrace{\frac{1}{\sqrt{x^2+y^2}}}_{\to 0} \longrightarrow 0$$

Thus, $u(x,y) \to 0$. Similarly, $v(x,y) \to 0$.

Ex: $\lim_{z\to\infty} e^z = ?$



Let z travel on the path $\gamma: z(t) = i+t$, $t \in \mathbb{R}$. This is the horizontal line passing through i.

$$e^z = e^{i+t} = e^t \operatorname{cis} 1$$

$$|e^z| = e^t \longrightarrow \begin{cases} 0 & \text{if } t \to -\infty \\ \infty & \text{if } t \to \infty \end{cases}$$

Thus, e^z has different limits as z goes to infinity westbound and eastbound. The limit $\lim_{z\to\infty} e^z$ doesn't exist.

* Derivative of function $f: G \subset \mathbb{C} \to \mathbb{C}$

We defined derivative of functions $\gamma: I \subset \mathbb{R} \to \mathbb{C}$ as

$$\gamma'(t_0) = \lim_{t\to t_0} \frac{\gamma(t)-\gamma(t_0)}{t-t_0}$$

This is simply derivative componentwise: $\gamma'(t) = x'(t) + iy'(t)$

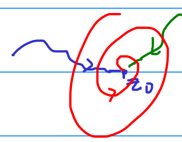
This simplicity is due to the fact that t is a real variable. There are not many ways for t to approach t_0.

Derivative of function $f: G \subset \mathbb{C} \to \mathbb{C}$ at z_0 is defined as

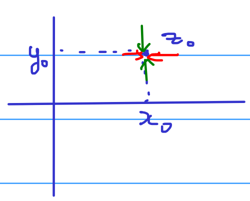
$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad (*)$$

This looks just the same as derivative of functions with real variable, but it is in fact quite strange because of division by complex number. In particular, $f'(z_0)$ no longer represents the rate of change of f in the usual sense. It represents something else. We will see that it represents how angles are changed after transformation.

Because there are many paths for z to approach z_0, and the limit $(*)$ is supposed to be the same for all chosen path, it is very hard for complex function to be differentiable at a point.



How to check if a function $f = u + iv$ is differentiable at z_0?



On the horizontal direction,

$$f'(z_0) = \lim_{x \to x_0} \frac{u(x, y_0) + i v(x, y_0) - (u(x_0, y_0) + i v(x_0, y_0))}{x - x_0}$$

$$= \frac{\partial u}{\partial x}(z_0) + i \frac{\partial v}{\partial x}(z_0)$$

On the vertical direction,

$$f'(z_0) = \lim_{y \to y_0} \frac{(u + iv) - (u_0 + iv_0)}{iy - iy_0} = -i \frac{\partial u}{\partial y}(z_0) + \frac{\partial v}{\partial y}(z_0)$$

For f to be differentiable at z_0, it is necessary that

$$\underbrace{\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}}_{\text{these are called Cauchy-Riemann equations}} \quad \text{at } z_0$$