Lecture 15 (5/13/2019)

Recall:

f is differentiable at z_0 $\Rightarrow$ $\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$ Consistency of limits as $z \to z_0$ horizontally and vertically.

How about the converse?

Thm: (Cauchy–Riemann)

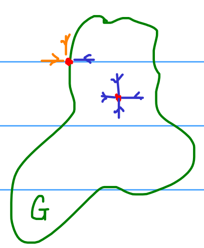
$f: G \subset \mathbb{C} \to \mathbb{C}$, $f = u + iv$, and $z_0 \in \mathring{G}$. If

- $\partial_x u, \partial_y u, \partial_x v, \partial_y v$ are cont. at z_0,
  (In other words, u and v are continuously differentiable at z_0.)
- u and v satisfy C-R eqs. at z_0

then f is differentiable at z_0. Moreover,

$$f'(z) = \frac{df}{dz} \overset{x\text{-direction}}{=\!=} \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \overset{y\text{-direction}}{=\!=} \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

If f is diff. in a disk $D_r(z_0)$, it is said to be holomorphic at z_0.
If f is diff. everywhere in $\mathbb{C}$, it is said to be an entire function.



We don't consider derivative at boundary points $z_0 \in \partial G$ because some directions to approach z_0 may be forbidden (going outside or starting from outside of G).

Ex: $f(z) = \bar{z} \operatorname{Re} z = (x - iy)x = x^2 - ixy$

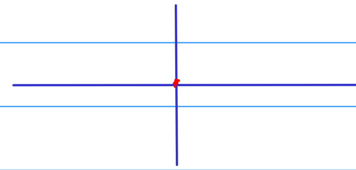
$u(x,y) = x^2$

$v(x,y) = -xy$

$\partial_x u = 2x$, $\partial_y u = 0$, $\partial_x v = -y$, $\partial_y v = -x$

C-R eqs: $\begin{cases} 2x = -x \\ 0 = y \end{cases} \rightsquigarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

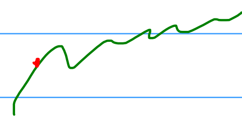
f is differentiable at $z = 0$, and nowhere else.



f is not holomorphic anywhere.

$$f'(0) = \partial_x u(0,0) + i\,\partial_x v(0,0) = 0.$$

Note: it's hard to construct a real-varialled function, say $f: \mathbb{R} \to \mathbb{R}$, that is nowhere differentiable but at one point. Such a function is not even intuitive. However, it is easy to point out an example for complex-varialled function (as above).



Ex: $f(z) = e^z = \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$

u and v are continuously differentiable

$\partial_x u = e^x \cos y, \qquad \partial_y v = e^x \cos y,$

$\partial_x v = e^x \sin y, \qquad \partial_y u = -e^x \sin y.$

$\rightsquigarrow$ C-R eqs. are satisfied.

$$(e^z)' = \frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y = e^z$$

e^z is an entire function.