Lecture 16 (5/8/2019)

We considered examples of complex functions being differentiable only on a curve and nowhere else. This region of differentiability is too small to do calculus. A function holomorphic at some point (differentiable on at least a disk) is a more prefered object to study calculus on.

So far, we know 2 methods to check differentiability and compute derivatives: using definition or C-R equations.

Ex: $f(z) = z^2$

1^{st} method: use definition

$$\frac{z^2 - z_0^2}{z - z_0} = \frac{(z-z_0)(z+z_0)}{z - z_0} = z + z_0 \longrightarrow 2z_0 \quad \text{as } z \to z_0$$

Thus, $(z^2)' = 2z$.

2^{nd} method: use C-R's theorem

$$z^2 = \underbrace{x^2 - y^2}_{u} + i \underbrace{2xy}_{v}$$

$$\left.\begin{array}{ll} \partial_x u = 2x, & \partial_x v = 2y \\ \partial_y u = -2y, & \partial_y v = 2x \end{array}\right\} \text{ C-R eqs. are satisfied}$$

$$(z^2)' = \partial_x u + i\partial_x v = 2x + i2y = 2z.$$

There are cleaner ways to check differentiability and compute derivatives.

* Laws of derivatives

Sum: $(f(z) + g(z))' = f'(z) + g'(z)$

Product (Leibniz's rule): $(f(z)g(z))' = f'(z)g(z) + f(z)g'(z)$

Quotient: $$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$

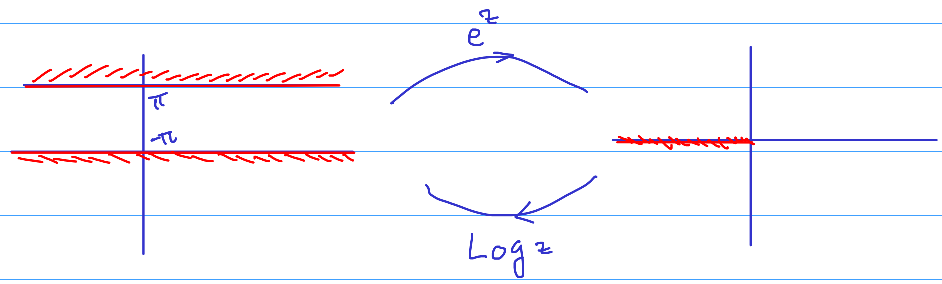
Composition (chain rule): $[f(g(z))]' = f'(g(z))\, g'(z)$

Ex: $(z^n)' = n z^{n-1}$, where $n = 1, 2, 3 \ldots$ (use product rule)

Ex: $\left(\frac{1}{z}\right)' = -\frac{1}{z^2} \quad \forall z \in \mathbb{C}\setminus\{0\}$

Use quotient rule.

Ex:



$$(-\infty,\infty)\times(-\pi,\pi) \underset{\operatorname{Log} z}{\overset{e^z}{\rightleftarrows}} \mathbb{C}\setminus\mathbb{R}_{\leq 0}$$

$$e^{\operatorname{Log} z} = z$$

Take derivative of both sides: $(\operatorname{Log} z)' \underbrace{e^{\operatorname{Log} z}}_{z} = 1$

Principal logarithm is holomorphic on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$ and $(\operatorname{Log} z)' = \frac{1}{z}$.

Ex: $f(z) = \sqrt{z}$ (principal branch)

By definition, $f(z) = e^{\frac{1}{2}\operatorname{Log} z}$.

f is holomorphic on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$ and

$$f'(z) = \frac{1}{2}(\operatorname{Log} z)' \, e^{\frac{1}{2}\operatorname{Log} z} = \frac{1}{2z}\sqrt{z} = \frac{1}{2\sqrt{z}}$$

principal branch

Observations:

* Derivative of logarithm is the same ($1/z$), regardless of the chosen branches. (Each branch differs from one another by a constant $k2\pi i$, whose derivative is zero).
* $(z^a)' = a\, z^{a-1}$

  same

  branch of logarithm

Constant functions

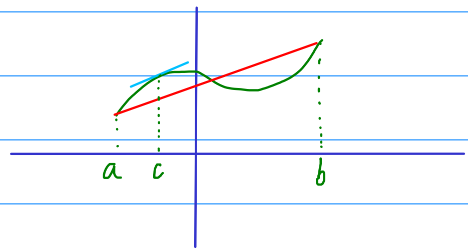
Suppose $f: G \subset \mathbb{C} \to \mathbb{C}$ satisfies $f'(z) = 0$ for all $z \in G$.
Is it possible to conclude that $f = \text{const}$ on G?
This is a basic question to ask before one considers antiderivatives.

$$\begin{cases} g' = f \\ h' = f \end{cases} \overset{?}{\Longrightarrow} \text{ g and h differ from each other by a constant.}$$

Recall how we answer this question in the case $f: I \subset \mathbb{R} \to \mathbb{R}$.

To show $f(a) = f(b)$ for any $x, y \in I$, we used Lagrange's theorem: if f is cont. on $[a,b]$ and differentiable on (a,b) then there exists $c \in (a,b)$ such that

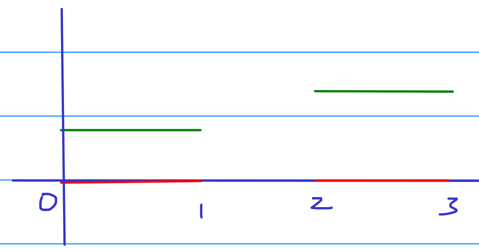
$$\underbrace{f'(c)}_{=0} = \frac{f(b) - f(a)}{b - a}$$



$\rightsquigarrow f(b) = f(a)$

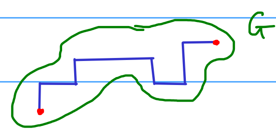
An important ingredient in the above argument is that f is differentiable on the entire interval (a,b). If I is not an interval, say $I = (0,1) \cup (2,3)$, then f wouldn't necessarily be constant on I. It is constant on $(0,1)$ and on $(2,3)$ separately.

connected components of I



Def: An open set $G \subset \mathbb{C}$ is said to be <u>connected</u> if any two points in G can be connected to each other by a path in G.

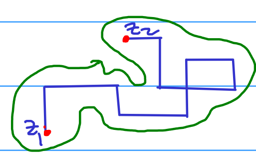
* Note: this path can be chosen to be a "rectangle" path:



Thm: Let $G \subset \mathbb{C}$ be an open connected subset. A function $f: G \to \mathbb{C}$ with $f'(z) = 0$ for all $z \in G$ must be constant.

why? Recall that $f'(z) = \partial_x u + i\partial_x v = \partial_y v - i\partial_y u$

Thus, $\partial_x u = \partial_y u = \partial_x v = \partial_y v = 0$ everywhere in G.



For $z_1, z_2 \in G$, there is a rectangle path in G that connects them. The value of f is the same on each straight segment of the path. Thus, $f(z_2) = f(z_1)$.

Examples of non-connected sets:

$G = G_1 \cup G_2$

$$f(z) = \begin{cases} a & \text{on } G_1 \\ b & \text{on } G_2 \end{cases}$$

$f'(z) = 0 \quad \forall z \in G$

but f is not constant on G.