Lecture 17 (5/10/2019)

If $F' = f$ on a connected set $G \subset \mathbb{C}$ then all antiderivatives of f on G is $F(z) + C$.

Why? Suppose $\tilde{F}$ is an antiderivative of f on G. Then

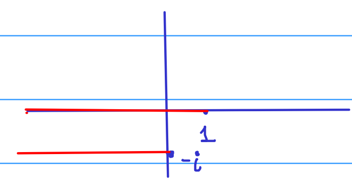
$$\tilde{F}' = f \quad \text{on } G$$

$$\rightsquigarrow (\tilde{F} - F)' = 0 \quad \text{on } G$$

$$\rightsquigarrow \tilde{F} - F = \text{const} \quad \text{on } G \quad \text{since } G \text{ is connected.}$$

Ex: $f(z) = \sqrt{1-z} + \sqrt{i+z}$ (principal branch)

Domain of continuity / differentiability is $\mathbb{C} \setminus \{z : z \in \text{red lines}\}$



This region is open. Thus, f is holomorphic every where on it.

An antiderivative of f on this region is

$$F(z) = -\frac{2}{3}(1-z)^{3/2} + \frac{2}{3}(i+z)^{3/2}$$

principal branch

This region is connected. Thus, all antiderivatives of f are

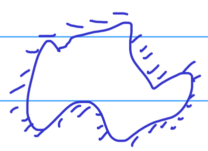
$$F(z) + C = -\frac{2}{3}(1-z)^{3/2} + \frac{2}{3}(i+z)^{3/2} + C$$

where C is a complex constant.

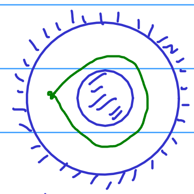
Thm: (to be proved next week)

If f is holomorphic on G and G is simply connected then it has an antiderivative on G.

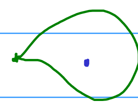
A simply-connected set is a connected set in which every closed path (loop) can be continuously contracted to a point. Intuitively, a simply-connected set is a connected set without "holes".



Simply-connected



not simply-connected (green curve can't contract to a point)



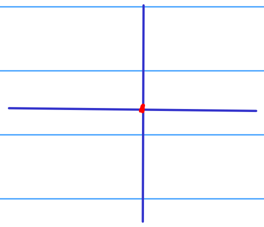
$\mathbb{C}\setminus\{0\}$

not simply-connected

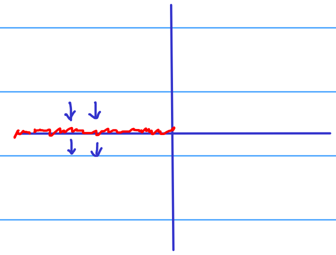
<u>Ex</u>: the function $\frac{1}{z}$ has no antiderivatives on $\mathbb{C}\setminus\{0\}$, although it is continuous on $\mathbb{C}\setminus\{0\}$. Why?

We know that $\operatorname{Log} z$ is an antiderivative of $1/z$ on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$.

$\mathbb{C}\setminus\{0\}$ is not simply connected. An antiderivative is not guaranteed to exist. In this example, it in fact doesn't exist.



jump of $-2\pi i$



$\mathbb{C}\setminus\mathbb{R}_{\leq 0}$ is simply connected. An antiderivative is guaranteed to exist.

Note that $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$ is a connected set. Any antiderivative of $1/z$ on $\mathbb{C}\setminus\{0\}$, if exists, must be equal to $\operatorname{Log} z + C$ on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$. The function $\operatorname{Log} z + C$ has a jump of $2\pi i$ accross the ray $\mathbb{R}_{\leq 0}$. There is no way to "fix" the function $\operatorname{Log} z + C$ on the ray $\mathbb{R}_{\leq 0}$ to make it continuous on there (not to say differentiable).

Mapping properties of holomorphic functions.

* Multiplication by complex number: $z \overset{\phi}{\mapsto} az$

Write $a = re^{i\theta}$.

$$\begin{tikzcd} z \arrow[r, mapsto, "\text{dilation}"] \arrow[rr, bend right, "\varphi"'] & rz \arrow[r, mapsto, "\text{rotation}"] & e^{i\theta} rz \end{tikzcd}$$

The multiplication by complex number $a = re^{i\theta}$ is obtained by stretching by factor r and then rotating by angle θ. The order of rotation and dilation can be reversed.

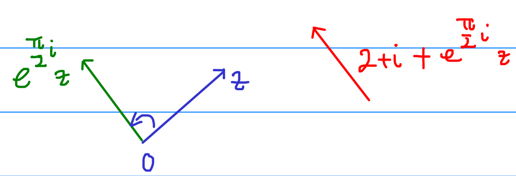
In complex standard form, $\varphi(z) = (r\cos\theta + ir\sin\theta)(x+iy) = \dots$
φ is a linear map. In matrix form,

$$\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \underbrace{r}_{\text{dilation}} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Ex: Rotation by $\pi/2$ about the origin followed by translation by vector $(2,1)$:

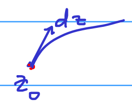
$$z \longrightarrow e^{i\frac{\pi}{2}} z \longrightarrow 2 + i + e^{i\frac{\pi}{2}} z$$

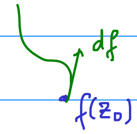
$$\varphi(z) = 2 + i + iz$$



In standard form: $\varphi(z) = 2 + i + i(x+iy) = 2 - y + i(1+x)$
which can be viewed as $\varphi(x,y) = (2-y, 1+x)$.

* General holomorphic map $f : G \subset \mathbb{C} \to \mathbb{C}$:





$$f'(z_0) \approx \frac{\Delta f(z_0)}{\Delta z_0}$$

$$\Delta f(z_0) \approx f'(z_0)\,\Delta z_0$$

(In precise form, $df = f'(z_0)\,dz$).

Write $f'(z_0) = re^{i\theta}$

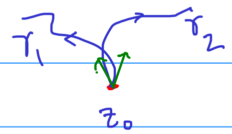
df is obtained by dilation dz by factor r and rotation by angle θ.

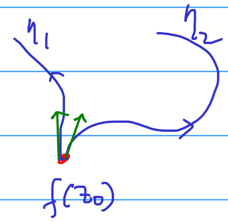
Let γ is a curve passing through z_0. Suppose $\gamma(0) = z_0$.

The image of γ under f is another curve: $\eta(t) = f(\gamma(t))$.

$$\underbrace{\eta'(0)}_{\substack{\text{tangent} \\ \text{vector of} \\ \eta \text{ at } f(z_0)}} = \underbrace{f'(\gamma(0))}_{re^{i\theta}}\ \underbrace{\gamma'(0)}_{\substack{\text{tangent vector} \\ \text{of } \gamma \text{ at } z_0.}}$$

$\eta'(0)$ is obtained from $\gamma'(0)$ by scaling with factor r and rotating by angle θ.





The angle between $\eta_1'(0)$ and $\eta_2'(0)$ is equal to the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$.

Def: A function $f: G \subset \mathbb{C} \to \mathbb{C}$ is said to be conformal if f preserves angles (both size and sign).

Thm: $f: G \subset \mathbb{C} \to \mathbb{C}$ is conformal if and only if

$$\begin{cases} f \text{ is differentiable on } G \text{ (in other words, holomorphic on } G) \\ f'(z) \neq 0 \quad \forall z \in G. \end{cases}$$