Lecture 18 (5/13/2019)

Recall the theorem:

[Function $f: G \subset \mathbb{C} \to \mathbb{C}$ is $\underbrace{\text{conformal}}_{\text{angle-preserving}}$ iff $\begin{cases} f \text{ is holomorphic on } G, \\ f'(z) \neq 0 \quad \forall z \in G \end{cases}$]

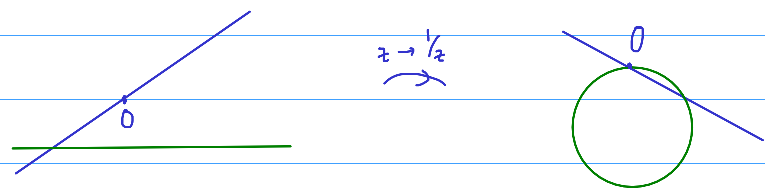
Ex: (Möbius transformation, also known as linear fractional transformation)

$$f(z) = \frac{az+b}{cz+d} \qquad (a, b, c, d \in \mathbb{C})$$

This is a combination of the following transformations:

- scaling
- rotation
- translation
- inversion $z \to 1/z$

The inversion maps circles and lines to either circles or lines.



line through origin $\mapsto$ line through origin,

line not through origin $\mapsto$ circle passing through origin.

* Line can be regarded as circle with radius equal ∞.

* Möbius transformation is closed under the family of lines and circles.

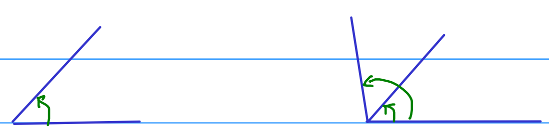
Ex:

$f(z) = z^2$, which can be identified as $f(x,y) = (x^2-y^2, 2xy)$.

This function is holomorphic on $\mathbb{C}$ and $f'(z) = 2z$.

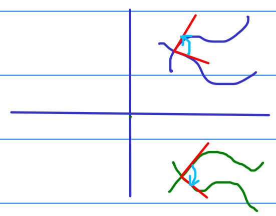
$f'(z) \neq 0$ on $\mathbb{C} \setminus \{0\}$

It is a conformal mapping on $\mathbb{C} \setminus \{0\}$. It is not conformal at 0 because angles are doubled in size.



Ex:

$f(z) = \bar{z}$ ...... $f(x,y) = (x,-y)$, which is not holomorphic anywhere (in fact, not differentiable anywhere).



This is reflection with respect to the y-axis. The size of angles is preserved, but the sign is reversed.

$\rightsquigarrow f(z)$ is not conformal map.

One can double check that f is not differentiable.

* Complex integration:

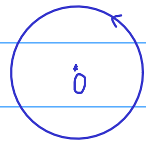
Review: how did we define complex differentiation? We mimick real diff.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad (*)$$

Why did we define real differentiation? to define tangent line, instant velocity, density, rate of change, etc. In other words, we were motivated by real life concepts. For complex-variabled functions, we follow a different methodology: we first formally define derivatives (from the fashion of real variables), then try to interpret what it means geometrically. For example, we have seen that if $f'(z_0)$ exists and nonzero the angles at z_0 are preserved. It maybe too difficult, and somewhat unnatural, to use angle-preserving property as a motivation to define complex derivative as in $(*)$.

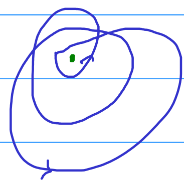
We will follow the same methodology for integration: first we try to define complex integrals mirroring real integrals. Then we try to interpret what it reflects, and how to apply it in meaning way. Here are some usage of complex integration which we will consider later:

- Detect singularity of a complex function. For example, if f is holomorphic on $D_r(0)$ then $\int_{C_r(0)} f(z)\,dz = 0$. But $\int_{C_r(0)} \frac{1}{z}\,dz = 2\pi i \neq 0$.



- Define winding number of a closed curve around a point. For example, if γ is a closed path wrapping around the unit circle n times then

$$\int_\gamma \frac{1}{z}\,dz = n2\pi i.$$



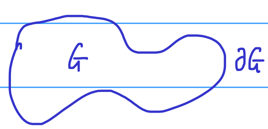
winding number $= 3$

Winding number is a useful notion in topology on 2D-plane.

- Prove the surprising result: if f is $\begin{cases} \text{holomorphic on } G \\ \text{continuous on } \overline{G} = G \cup \partial G \end{cases}$

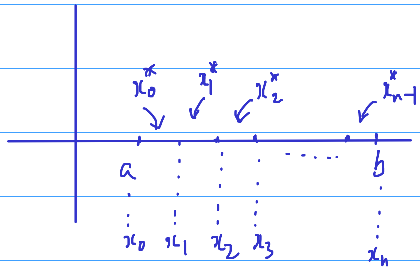
(called closure of G)

then the values of f in G is completely defined by the values of f on ∂G.



This result implies that holomorphic functions are quite rigid. What f does in the interior is determined by what it does on the boundary. This phenomenon doesn't happen for real-variabled functions, since one can always alter the values of f in G, while keeping values on ∂G, and still have a smooth function.

How did we define integration for real-variabled functions? Let $f: [a,b] \to \mathbb{R}$ be continuous.



First, partition the interval $[a,b]$ into subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

Secondly, sample a point from each interval, say x_k^* from $[x_k, x_{k+1}]$.

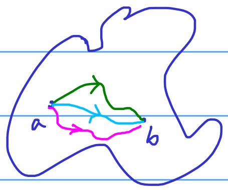
Thirdly, set up a Riemann sum

$$\sum_{k=0}^{n-1} f(x_k^*)\underbrace{(x_{k+1}-x_k)}_{\Delta x_k}$$

Finally, take the limit

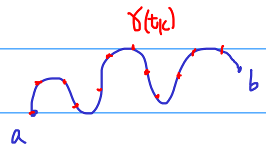
$$\lim_{n\to\infty}\sum_{k=0}^{n-1} f(x_k^*)(x_{k+1}-x_k) \;=:\; \int_a^b f(x)\,dx$$

For complex function $f: G\subset\mathbb{C}\to\mathbb{C}$ and $a, b\in G$, we want to make sense of the notation $\int_a^b f(z)\,dz$.



The "interval $[a,b]$" in real case is analogous to a path from a to b in complex case.

Since there are many ways to connect a to b, the notation $\int_a^b f(z)\,dz$ is not good. Instead, we will use the notation $\int_\gamma f(z)\,dz$, where γ is a specified path from a to b.



How to partition the path γ?

Look at the parametrization of γ: $\gamma(t)$, $\alpha\le t\le\beta$.

Partition the interval $[\alpha,\beta]$ by

$$\alpha = t_0 < t_1 < t_2 < \cdots < t_n = \beta$$

Establish Riemann sum:

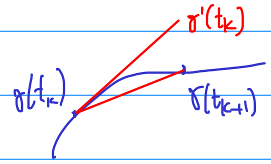
$$\sum_{k=0}^{n-1} f(\gamma(t_k))\underbrace{(\gamma(t_{k+1})-\gamma(t_k))}_{\text{difference between two consecutive points}}$$

Take limit:

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} f(\gamma(t_k))(\gamma(t_{k+1})-\gamma(t_k)) \;=:\; \int_\gamma f(z)\,dz$$

This definition is hard to do calculation with, unless one is assisted with Mathematica. How to simplify it?

$$\frac{\gamma(t_{k+1}) - \gamma(t_k)}{t_{k+1} - t_k} \approx \gamma'(t_k) \quad \text{since} \quad \gamma'(t_k) = \lim_{t \to t_k} \frac{\gamma(t) - \gamma(t_k)}{t - t_k}$$



Riemann sum $\approx \displaystyle\sum_{k=0}^{n-1} \underbrace{f(\gamma(t_k))\,\gamma'(t_k)}_{g(t_k)}\,(t_{k+1} - t_k)$

Take limit as $n \to \infty$:

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta \underbrace{f(\gamma(t))\,\gamma'(t)}_{g(t)}\,dt$$

How to interpret $\displaystyle\int_\alpha^\beta g(t)\,dt$? (Note that $g(t)$ is a complex number)

$$\int_\alpha^\beta g(t)\,dt = \underbrace{\int_\alpha^\beta \operatorname{Re} g(t)\,dt}_{} + i \underbrace{\int_\alpha^\beta \operatorname{Im} g(t)\,dt}_{}$$

these are usual (real) integrals.