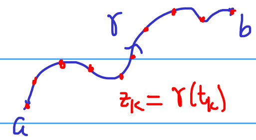
Lecture 19 (5/15/2019)



$$\underbrace{\int_\gamma f(z)\,dz}_{\text{integral of complex-valued function against complex variable}} := \lim_{n\to\infty} \sum_{k=0}^{n-1} f(z_k)(z_{k+1}-z_k)$$

$$= \int_\alpha^\beta \underbrace{f(\gamma(t))\,\gamma'(t)}_{g(t)}\,dt$$

Tip to remember: in the first integral, replace z by $\gamma(t)$ and use the change-of-variable rule $dz = \gamma'(t)\,dt$.

$$g(t) = g_1(t) + i g_2(t)$$

$$\underbrace{\int_\alpha^\beta g(t)\,dt}_{\text{integral of complex-valued function against real variable}} \quad \text{is interpreted as} \quad \underbrace{\int_\alpha^\beta g_1(t)\,dt + i\int_\alpha^\beta g_2(t)\,dt}_{\text{integrals of real-valued function against real variable}}$$

Ex: $\displaystyle\int_0^\pi e^{it}\,dt = ?$

$$\int_0^\pi e^{it}\,dt = \int_0^\pi (\cos t + i\sin t)\,dt = \sin t\Big|_0^\pi - i\cos t\Big|_0^\pi$$

$$= 0 - i(-1-1) = 2i$$

In most cases, one is able to find an antiderivative of g, say G.

$$G' = g$$

which means $G_1' = g_1$ and $G_2' = g_2$.

$$\int_\alpha^\beta g(t)\,dt = \int_\alpha^\beta (G_1' + iG_2')\,dt = \int_\alpha^\beta G_1'(t)\,dt + i\int_\alpha^\beta G_2'(t)\,dt$$

$$= \Big[G_1(t) + iG_2(t)\Big]_\alpha^\beta$$

$$= G(t)\Big|_\alpha^\beta.$$

Thus, $\displaystyle\int_\alpha^\beta g(t)\,dt = G(t)\Big|_\alpha^\beta$

which is of the same form as the Fundamental thm of Calc.

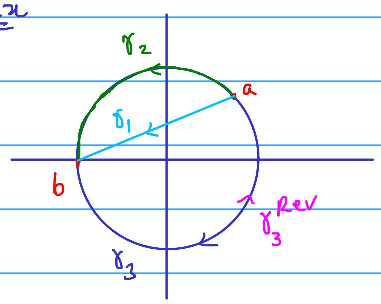
In the previous example, e^{it} has an antiderivative $\frac{1}{i}e^{it}$.

Why? $\quad \frac{d}{dz}\frac{1}{i}e^{iz} = e^{iz}$

When $z = t \in \mathbb{R}$, $\quad \frac{d}{dt}\frac{1}{i}e^{it} = e^{it}$.

Then $\quad \int_0^{\pi} e^{it}\,dt = \frac{1}{i}e^{it}\Big|_0^{\pi} = \frac{1}{i}\left(e^{i\pi} - e^{i0}\right) = \frac{1}{i}(-2) = 2i$

Ex



$f(z) = \frac{1}{z^2} = z^{-2}$

$a = e^{\frac{\pi}{4}i}$, $\quad b = -1$

$\gamma_1, \gamma_2, \gamma_3$ as indicated on the picture.

To find $\int_{\gamma_1} f(z)\,dz$, $\int_{\gamma_2} f(z)\,dz$, $\int_{\gamma_3} f(z)\,dz$, we first parametrize

$\gamma_1, \gamma_2, \gamma_3$:

- $\gamma_1(t) = a + t(b-a)$, $\quad 0 \le t \le 1$

  $\Rightarrow \gamma_1'(t) = b-a$

- $\gamma_2(t) = e^{it}$, $\quad \frac{\pi}{4} \le t \le \pi$

  $\Rightarrow \gamma_2'(t) = ie^{it}$

- $\gamma_3^{Rev}(t) = e^{it}$, $\quad -\pi \le t \le \pi/4 \quad \rightsquigarrow \quad \gamma_3(t) = \underbrace{e^{i(-\pi + \frac{\pi}{4} - t)}}_{e^{i(-\frac{3\pi}{4} - t)}}$, $\quad -\pi \le t \le \frac{\pi}{4}$

  $\Rightarrow \gamma_3'(t) = -ie^{-i(\frac{3\pi}{4}+t)}$

$I_1 = \int_{\gamma_1} f(z)\,dz = \int_0^1 \frac{1}{(\bar{a} + t(\bar{b}-\bar{a}))^2}(b-a)\,dt \qquad (*)$

The integrand has antiderivative

$$\frac{b-a}{\bar{b}-\bar{a}} \quad \frac{-1}{\bar{a}+t(\bar{b}-\bar{a})}$$

Thus,

$$I_1 = \frac{b-a}{\bar{b}-\bar{a}} \quad \frac{-1}{\bar{a}+t(\bar{b}-\bar{a})}\Bigg|_0^1 = \frac{b-a}{\bar{b}-\bar{a}}\left(-\frac{1}{\bar{b}}+\frac{1}{\bar{a}}\right)$$

$$= \frac{b-a}{\bar{b}\,\bar{a}} = \frac{\sqrt{2}}{2}+1-i\frac{\sqrt{2}}{2}$$

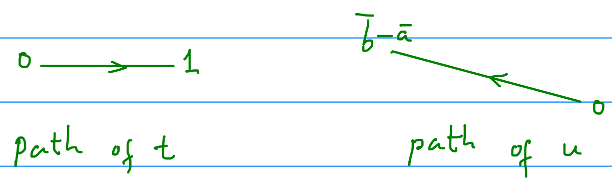
<u>Warning</u>: to compute (*), it is tempting to use substitution

$$u = t(\bar{b}-\bar{a}), \qquad du = (\bar{b}-\bar{a})dt$$

Then

$$(*) = \int_0^{\bar{b}-\bar{a}} \frac{1}{(\bar{a}+u)^2} \frac{b-a}{\bar{b}-\bar{a}}\, du \qquad (**)$$

However, one can see that $\bar{b}-\bar{a} \notin \mathbb{R}$. The integral (**) is not an integral over real variable, since $u \notin \mathbb{R}$. The path from 0 to $\bar{b}-\bar{a}$ must be understood as the image of the path $0 \longrightarrow 1$ of t under the mapping $t \mapsto \underbrace{t(\bar{b}-\bar{a})}_{\text{scale by } |\bar{b}-\bar{a}| \text{ and rotate by } \operatorname{Arg}(\bar{b}-\bar{a})}$.



In short, one has to be careful when attempting to use substitution. The (real) interval of integration will change into a complex path. In most cases, this complicates the problem, not simplifying it.

$$I_2 = \int_{\pi/4}^{\pi} \frac{1}{(e^{-it})^2}\, ie^{it}\,dt = \int_{\pi/4}^{\pi} i e^{3it}\,dt = i\frac{1}{3i}e^{3it}\Big|_{\pi/4}^{\pi}$$

$$= \frac{1}{2i}\Big(\underbrace{e^{i3\pi}}_{-1} - \underbrace{e^{i\frac{3\pi}{4}}}_{-\sqrt{2}/2+i\sqrt{2}/2}\Big)$$

$$= \dots$$

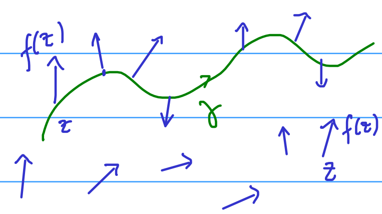
$$I_3 = \int_{-\pi}^{\pi/4} \frac{1}{e^{(\frac{3\pi}{4}+t)2t}} (-i)\, e^{(-\frac{3\pi}{4}-t)i}\, dt = \int_{-\pi}^{\pi/4} -i\, e^{-3(\frac{3\pi}{4}+t)i}\, dt = -i \frac{1}{-3i} e^{-3(\frac{3\pi}{4}+t)i} \Big|_{-\pi}^{\pi/4}$$

$$= \cdots$$

Now that we have defined complex integrals (in the same fashion as real integals), let's try to interpret its meaning geometrically.

A map $f : G \subset \mathbb{C} \to G$ can be viewed as

- function : takes a number to a number,
- 2D transformation : takes a point to a point,
- vector field : takes a point to a vector (in other words, assigns a vector to a point).



$$f(z) = \underbrace{u(z) + i\,v(z)}_{\text{real-valued}} \equiv (u(z), v(z))$$

Note that

$$\underbrace{\int_\gamma f(z)\, dz}_{\text{complex integral}} \neq \underbrace{\int_\gamma f(z) \cdot d\vec{s}}_{\text{line integral}}$$

because $f(z)\,dz$ symbolizes $f(z_k)\,(z_{k+1} - z_k)$, whereas
complex multiplication

$f(z) \cdot d\vec{s}$ symbolizes $(u(z), v(z)) \cdot d\vec{s}$.
dot product

Let's look closer to each integral:

$$dz = dx + i\,dy, \qquad d\vec{s} = (dx, dy)$$

$$\int_\gamma f(z)\cdot d\vec{s} = \int_\gamma (u,v)\cdot(dx,dy) = \int_\gamma u\,dx + v\,dy$$

$$= W[f,\gamma]$$

This is the work done by force field f along path γ.

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+i\,dy) = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy - v\,dx)$$

$$= W[\bar{f},\gamma] + i\,F[\bar{f},\gamma]$$

$W[\bar{f},\gamma]$ is the work done by the conjugate field $\bar{f} = (u,-v)$ along γ,

$F[\bar{f},\gamma]$ is the flux of $\bar{f}$ across γ.