Lecture 2 (4/3/2019)

Reflection: calculus of single real variable consists of differential calc. and integral calc. Both are founded on:

- algebraic structure of real numbers,
- the notion of limits.

Derivative: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Integral: $$\int_a^b f(x)\,dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k)$$

To establish calculus for complex variable, one needs algebraic properties (addition, subtraction, multiplication, division) of complex numbers and the notion of limits.

* Algebraic properties of complex numbers:

The alg. properties (of real numbers) that are necessary to define derivative and integral are:

- Addition: if $a, b \in \mathbb{R}$ then $a+b \in \mathbb{R}$
  - Neutral element: $a+0 = 0+a = a$
  - Additive inverse: for each $a \in \mathbb{R}$ there is a unique number (called $-a$) such that $a+(-a) = (-a)+a = 0$
  - Commutativity: $a+b = b+a$ (order doesn't matter)
  - Associativity: $a+(b+c) = (a+b)+c$ (grouping differently gives the same result)
- Multiplication: if $a, b \in \mathbb{R}$ then $ab \in \mathbb{R}$
  - Identity element: $1a = a1 = a$
  - Unit element: for each $a \in \mathbb{R} \setminus \{0\}$ there is a unique number (called a^{-1}) such that $aa^{-1} = a^{-1}a = 1$
  - Commutativity: $ab = ba$
  - Associativity: $a(bc) = (ab)c$
- Distributivity: $(a+b)c = ac + bc$

In other words, real numbers form a <u>field</u>.

$\mathbb{R}$ has another structure: order, which it inherits from $\mathbb{N}$ (natural numbers)

as a consequence $x^2 \geq 0$ for all x

The equation $x^2+1=0$ has no roots in $\mathbb{R}$.

Note: this phenomenon doesn't come from the field properties of $\mathbb{R}$. For example,

$$\left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$ satisfies all field properties

Neutral element: $\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{"0"}}$

Unit element: $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{"1"}}$. Note that $\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{x} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{x} = \underbrace{-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{"}-1\text{"}}$

About impossibilities:

- Division by 0 $\rightsquigarrow$ forbidden by field properties.

  If $1/0 = a \in \mathbb{R}$ then for any $x \in \mathbb{R}$,

  $$x = x1 = x(0a) = (x0)a = 0a = 1$$

  $\rightsquigarrow$ every number would be equal to 1 (contradiction)

- How impossibilities are fixed by expanding the set of numbers:

$$\mathbb{N} \xrightarrow{\text{adopt "}-1\text{"}} \mathbb{Z} \xrightarrow{\text{adopt "}\frac{1}{2}\text{", "}\frac{2}{3}\text{"} \dots} \mathbb{Q} \xrightarrow{\text{"}\sqrt{2}\text{", "}\sin 1\text{", } \dots} \mathbb{R} \xrightarrow{\text{adopt "}\sqrt{-1}\text{"}} \mathbb{C}$$

$2-5$ forbidden (impossible to subtract a larger number from a smaller number)	$\frac{1}{2}$ forb. (impossible to perform division for many pairs of numbers)	completeness issue: $1^2<2$, $2^2>2$ but there's no $x \in \mathbb{R}$ s.t. $x^2=2$	square root of neg. numbers forbidden	algebraically closed (n'th degree poly. has n roots)

* Notion of limits:

$$\lim_{x \to a} f(x) = b$$

$\Updownarrow$

For each $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x-a| < \delta$ then $|f(x) - b| < \varepsilon$.

Notice that the order in $\mathbb{R}$ was replaced by the notion of distance.

In short, one needs

$$\begin{cases} \text{field structure} \\ \text{notion of distance} \end{cases}$$

on complex numbers to be able to establish calculus in a way similar to that of real variable.

Complex number: one adopts a new number $\underbrace{\sqrt{-1}}$.

but this is not a good notation because $\sqrt{-1}$ actually represents two numbers.

$$\sqrt{-1} = \{i, -i\}$$

imaginary unit

Roughly speaking, $\mathbb{C}$ (complex numbers) consists of $\mathbb{R}$ (real numbers) and i and all other numbers that can be generated through addition and multiplication.

Ex:

$1+2i$ (better notation for $1+2\sqrt{-1}$)

$$(2+3i)^2 = (2+3i)(2+3i) = 4+6i+6i+9i^2$$
$$= 4+12i+9(-1)$$
$$= -5+12i$$

$\mathbb{C} = \{a+bi : a,b \in \mathbb{R}\}$

Neutral element : $0+0i$ $(=0)$

Unit element : $1+0i$ $(=1)$

$$z = \underbrace{a}_{Re(z)} + i\underbrace{b}_{Im(z)} \quad \text{..... standard form}$$

$z^{-1} = ?$, put $z^{-1} = x+iy$

Want: $(x+iy)(a+ib) = 1$

$$\begin{cases} ax - by = 1 \\ bx + ay = 0 \end{cases} \rightsquigarrow \text{system of 2 eqs., 2 unknowns.}$$

$$x = \frac{\begin{vmatrix} 1 & -b \\ 0 & a \end{vmatrix}}{\begin{vmatrix} a & -b \\ b & a \end{vmatrix}} = \frac{a}{a^2+b^2}, \quad y = \cdots = \frac{-b}{a^2+b^2}$$

$z^2 = (a+bi)^2 = a^2-b^2+2ab\,i$

$\bar{z} = a-bi$ ...... complex conjugate of $z = a+bi$

$$\frac{a+bi}{\underset{\sim}{c+di}} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd+i(bc-ad)}{c^2+d^2}$$

nonstandard form

$$= \underbrace{\frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2}}_{\text{standard form}}$$