Lecture 20 (5/17/2015)

* Properties of complex integrals:

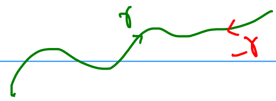
- For $c_1, c_2 \in \mathbb{C}$, f and g continuous functions,

$$\int_\gamma \left(c_1 f(z) + c_2 g(z)\right) dz = c_1 \int_\gamma f(z)\,dz + c_2 \int_\gamma g(z)\,dz$$

- $$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz \qquad (*)$$

where $-\gamma$ is the opposite / reversed path of γ : $(-\gamma)(t) = \gamma(\alpha+\beta-t)$

Why? $(-\gamma)'(t) = -\gamma'(\alpha+\beta-t)$.

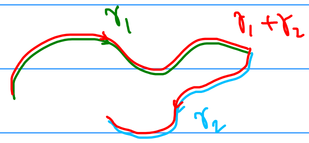


Property $(*)$ is analogous to $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ for real functions f.

- $$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz \qquad (**)$$

where $\gamma_1+\gamma_2$ is continuation of γ_1 by γ_2.

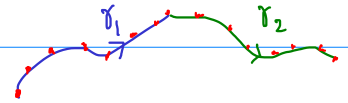
Note: the textbook uses notation $\gamma_1\gamma_2$ instead of $\gamma_1+\gamma_2$.



For example, if $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ are paths such that $\gamma_1(1) = \gamma_2(0)$ then one can parametrize the path $\gamma = \gamma_1 + \gamma_2$ as follows:

$$\gamma(t) = \begin{cases} \gamma_1(2t), & 0 \le t \le 1/2 \\ \gamma_2(2t-1), & 1/2 \le t \le 1. \end{cases}$$

How to explain $(**)$?



$$\sum_{k=0}^{n-1} f(z_k)(z_{k+1}-z_k) = \sum_{\text{on } \gamma_1} \cdots + \sum_{\text{on } \gamma_2} \cdots$$

$$\downarrow n\to\infty \qquad\qquad \downarrow n\to\infty \qquad\qquad \downarrow n\to\infty$$

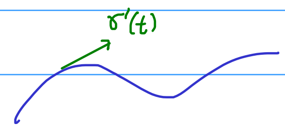
$$\int_\gamma f\,dz \qquad\qquad \int_{\gamma_1} f\,dz \qquad\qquad \int_{\gamma_2} f\,dz$$

Prop. (**) is analogous to $\int_a^b f\,dx + \int_b^c f\,dx = \int_a^c f\,dx$ for real functions f.

Some remarks:

① One doesn't need $\gamma'(t) \neq 0$ (path being regular) to define the integral $\int_\gamma f(z)\,dz$

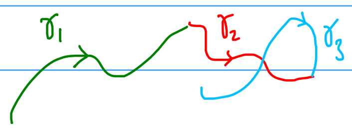
The condition $\gamma'(t) \neq 0$ becomes relevant when one considers tangent vectors.



If $\gamma'(t) = 0$ for some t then $\gamma'(t)$ fails to be a meaningful tangent vector of γ.

② In the definition of $\int_\gamma f\,dz$, γ is allowed to be piecewise smooth

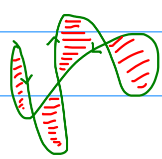
(not necessarily smooth as a whole). Also, γ is allowed to intersect itself.



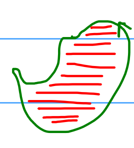
$$\int_\gamma f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz - \int_{\gamma_3} f\,dz$$

$\gamma = \gamma_1 + \gamma_2 - \gamma_3$

orientation changes back and forth from positive to negative



non-simple path

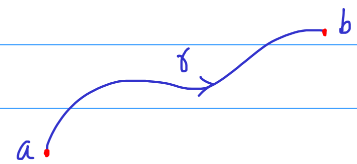


simple path, positively oriented

In some situations, simple paths (path that doesn't intersect itself, possibly except at the start and ending points) are more convenient to work with because it can be oriented positively or negatively.

* If $f: G \subset \mathbb{C} \to \mathbb{C}$ has an antiderivative F and $\gamma \subset G$ then $\int_\gamma f(z)\,dz$ only depends on the ending points. Specifically,

$$\int_\gamma f(z)\,dz = F(b) - F(a)$$



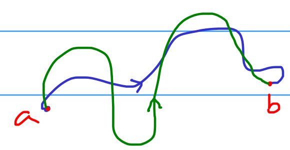
To see later: if $f: G \subset \mathbb{C} \to \mathbb{C}$ is holomorphic and G is simply-conn. then f has an antiderivative on G.

Why?

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\,\gamma'(t)\,dt = \int_\alpha^\beta \underbrace{F'(\gamma(t))\,\gamma'(t)}_{[F(\gamma(t))]'}\,dt$$

$$= F(\gamma(\beta)) - F(\gamma(\alpha)) \quad \text{(Fund. Thm. of Calc.)}$$

$$= F(b) - F(a)$$

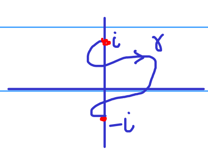


Ex: γ is a path connecting 1 to $1+2i$.

$$\int_\gamma (z^2+z)\,dz = \left(\frac{z^3}{3} + \frac{z^2}{2}\right)\Bigg|_1^{1+2i} = \frac{(1+2i)^3}{3} + \frac{(1+2i)^2}{2} - \left(\frac{1}{3} + \frac{1}{2}\right)$$

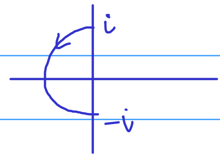
$$= \cdots$$

Ex: γ is a path connecting i to $-i$, lying in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.



$$\int_\gamma \frac{1}{z}\,dz = \operatorname{Log} z \Big|_i^{-i} = \operatorname{Log}(-i) - \operatorname{Log} i = \cdots$$

Ex: γ is the left half of the unit circle connecting i to $-i$.

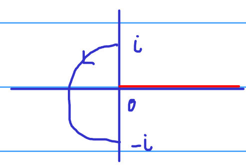


$$\int_\gamma \frac{1}{z}\,dz = ?$$

We see that γ cuts through the line $\mathbb{R}_{\le 0}$, where $\operatorname{Log} z$ fails to be antiderivative of $1/z$.

There are two methods to deal with this situation:

① Find a different antiderivative of $1/z$ such that γ lies entirely in its domain. For example, choose a branch for the argument as $\arg z \in (0, 2\pi)$.



Corresponding branch of logarithm:

$$\mathcal{L}og\, z = \ln|z| + i \arg z$$

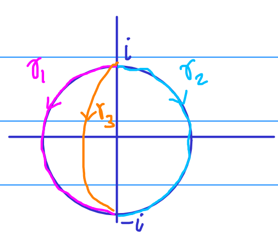
which is an antiderivative of $1/z$ on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

$$\int_\gamma \frac{1}{z}\,dz = \mathcal{L}og\, z \Big|_i^{-i} = \mathcal{L}og(-i) - \mathcal{L}og(i)$$

$$= \ln|-i| + i \arg(-i) - (\ln|i| + i\arg(i))$$

$$= 0 + i\frac{3\pi}{2} - \left(0 + i\frac{\pi}{2}\right)$$

$$= i\pi$$



* Comments: $\displaystyle\int_{\gamma_1} \frac{1}{z}\,dz = i\pi \neq -i\pi = \int_{\gamma_2} \frac{1}{z}\,dz$

Integrals of f over γ_1 and γ_2 are different because γ_1 and γ_2 don't lie in the same region where $1/z$ has an antiderivative. In fact, any branch cut of the logarithm, if wanting to avoid γ_1, must intersect γ_2, and vice versa.

$\displaystyle\int_{\gamma_1} f\,dz = \int_{\gamma_3} f\,dz$ because γ_1 and γ_3 lie in the same region where $1/z$ has an antider.

② Use definition of complex integral: start with parametrizing γ

$$\gamma(t) = e^{it}, \quad \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$$

$$\int_\gamma \frac{1}{z}\, dt = \int_{\pi/2}^{3\pi/2} \frac{1}{e^{it}}\, i e^{it}\, dt = i\pi$$