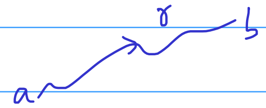
Lecture 21 (5/20/2019)

We know that if a function f has an antiderivative on a region G and $\gamma \subset G$ then $\int_\gamma f\,dz$ only depends on the endpoints of γ. More specifically,

$$\int_\gamma f\,dz = F(b) - F(a)$$



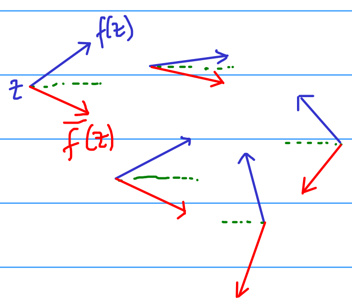
A useful consequence is the following:

Let $f: G \subset \mathbb{C} \to \mathbb{C}$ and γ be a closed path (loop) contained in G. Suppose f has an antiderivative on G. Then

$$\int_\gamma f\,dz = 0$$

The property of "integral over a loop equal to zero" is a desirable feature of a complex function. If a function has an antiderivative, we know that it has this feature. However, it is sometimes hard to check if an antiderivative exists. It's easier to check if derivative exists, for example using Cauchy-Riemann equations. Later in this lecture, we will state a result analogous to the above theorem, but for holomorphic functions (known as Cauchy-Goursat theorem).

* Geometric interpretation of complex integrals:



$f(z) = u(z) + iv(z) = (u(z), v(z))$ can be thought as a vector field: to each point z is assigned a vector $f(z)$.

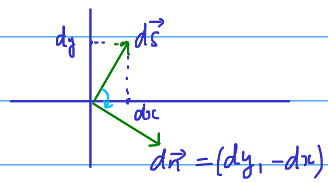
$\bar{f}(z) = u(z) - iv(z) = (u(z), -v(z))$ is the conjugate vector field, or Pólya vector field of f.

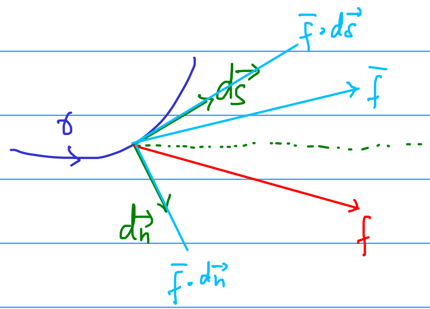
$$\int_\gamma f\cdot d\vec{s} = \int_\gamma (u,v)\cdot(dx,dy) = W[f,\gamma]$$ .... work done by f along γ.

$$\int_\gamma f(z)\,dz = \int (u+iv)(dx+i\,dy) = \underbrace{\int_\gamma u\,dx - v\,dy}_{(u,-v)\cdot(dx,dy)} + i\underbrace{\int_\gamma u\,dy - v\,dx}_{(u,-v)\cdot(dy,-dx)}$$

$$= \bar{f}\cdot d\vec{s} \qquad\qquad = \bar{f}\cdot d\vec{n}$$

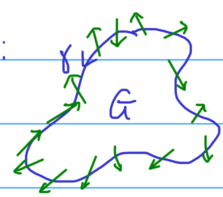
$$= W[\bar{f},\gamma] + i\,F[\bar{f},\gamma]$$





$d\vec{n} = (dy,-dx)$ is obtained by rotating $d\vec{s}$ clockwise.

Suppose γ is a simple closed curve, possitively oriented:
doesn't intersect itself



- Green's theorem: $W[\bar{f},\gamma] = \int_G \operatorname{curl}\bar{f}\,dA = \int_G (-\partial_x v - \partial_y u)\,dA$

Recall: $\operatorname{curl}(P,Q) = \partial_x Q - \partial_y P$

curl of a 2D-vector field is a scalar function

$$\operatorname{curl}(P,Q,R) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ P & Q & R \end{vmatrix}$$

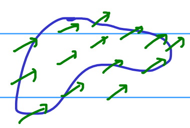
curl of a 3D-vector field is a 3D-vector field.

- Divergence theorem:

$$F[\bar{f},\gamma] = \int_G \operatorname{div}\bar{f}\, dA = \int_G (\partial_x u - \partial_y v)\, dA$$

* Consequence: $$W[f,\gamma] + iF[f,\gamma] = \int_\gamma \bar{f}(z)\, dz$$

Ex:



Constant vector field $g(x,y) = (a_1, a_2)$, which can be considered as constant complex function

$$g(z) = a = a_1 + i a_2.$$

$$\bar{g}(z) = \bar{a}, \text{ whose antiderivative is } \bar{a} z.$$

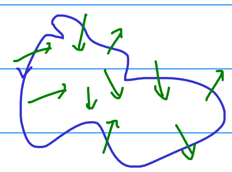
For any loop γ, $W[g,\gamma] = F[g,\gamma] = 0$ because $\int_\gamma \bar{g}\, dz = 0$.

Ex: Consider vector field $f(x,y) = (x^2, xy)$.

It can be viewed as complex function $f(z) = x^2 + ixy$.

Pólya vector field: $\bar{f}(z) = x^2 - ixy$.

$$W[f,\gamma] + iF[f,\gamma] = \int_\gamma (x^2 - ixy)\, dz$$



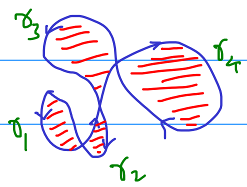
Thm (Cauchy-Goursat):

If f is holomorphic on the region G enclosed by a simple $\overbrace{\text{closed path}}^{\text{loop}}$ γ and continuous on $\bar{G} = G \cup \gamma$ then $\int_\gamma f\, dz = 0$.

Why? If γ is a simple path, use Cauchy-Riemann eqs.

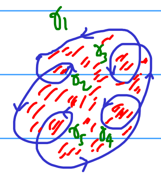
$$\int_\gamma f(z)\, dz = \int_G (\underbrace{-\partial_x v - \partial_y u}_{=0})\, dA + i \int_G (\underbrace{\partial_x u - \partial_y v}_{=0})\, dA = 0$$

If γ is not a simple path, one decomposes it into simple paths.



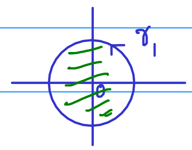
$$\int_\gamma f\,dz = \underbrace{\int_{\gamma_1} f\,dz}_{0} + \underbrace{\int_{\gamma_2} f\,dz}_{0} + \underbrace{\int_{\gamma_3} f\,dz}_{0} + \underbrace{\int_{\gamma_4} f\,dz}_{0}$$

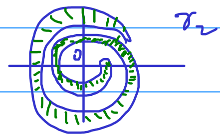
since each of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ is simple path.



$$\int_\gamma = \underbrace{\int_{\gamma_1}}_{0} + \underbrace{\int_{\gamma_2}}_{0} + \underbrace{\int_{\gamma_3}}_{0} + \underbrace{\int_{\gamma_4}}_{0} + \underbrace{\int_{\gamma_5}}_{0} = 0$$

Ex: Function $f(z) = 1/z$ is not holomorphic at $z = 0$.





Cauchy-Goursat's thm
not applicable

$$\int_{\gamma_2} \frac{1}{z}\,dz = 0$$

because $1/z$ is holomorphic in the region enclosed by γ_2.