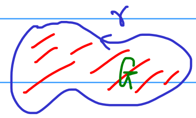
Lecture 22 (5/22/2019)

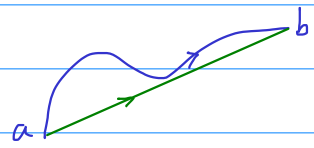
Recall Cauchy-Goursat thm: given a simple loop γ, let G be the region enclosed by γ. If f is holomorphic in G and continuous on $G \cup \gamma$ then $\int_\gamma f\,dz = 0$.



This theorem states for an individual curve. Now let's ask the following question:

[Consider a function $f: G \subset \mathbb{C} \to \mathbb{C}$. Under what condition of f do complex integrals $\int_\gamma f(z)\,dz$ only depend on the endpoints of γ, for any curve $\gamma \subset G$?]

Any answer to this question of course has computational value: one can replace γ by another curve with the same endpoints and having simpler parametrization.



We know that if f has an antiderivative, then $\int_\gamma f\,dz$ only depends on the endpoints of γ. How about if f is holomorphic on G?

Thm:

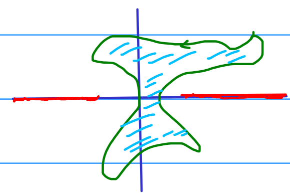
Let $f: G \subset \mathbb{C} \to \mathbb{C}$ is holomorphic and G is simply connected region then $\int_\gamma f(z)\,dz$ only depends on the endpoints of γ.

Recall: G being simply-connected means that every loop in G can contract continuously to a point.

Why?

Any loop $\gamma \subset G$ (simply connected) encloses a subregion G' of G. Since f is holomorphic in G, so is it on G'. Thus, $\int_\gamma f\,dz = 0$.

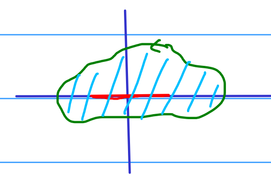
The shaded region is entirely contained in G



$G = \mathbb{C} \setminus \{z \in \mathbb{R}, |z| \geq 1\}$

simply connected

The shaded region is not entirely contained in G.



$G = \mathbb{C} \setminus \{z \in \mathbb{R}, |z| \leq 1\}$

not simply connected

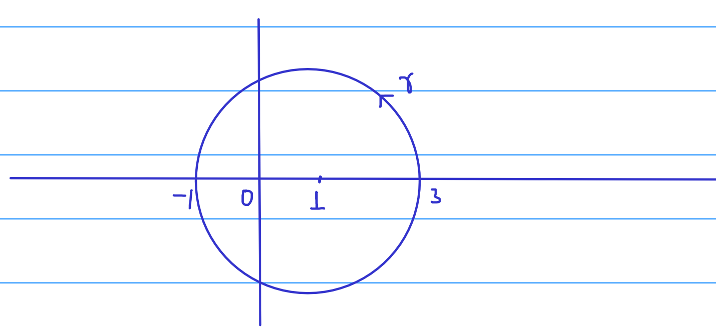
* Conclusion:

$$\left.\begin{array}{c}\text{Having antiderivative} \\ \text{OR} \\ \text{Holomorphic + simply conn.}\end{array}\right\} \Rightarrow \int_\gamma f\,dz = 0 \quad \forall \text{ loop } \gamma \subset G$$

Ex:

$$\int_\gamma \frac{1}{z}\,dz$$

where γ is the circle $C_1(2)$ positively oriented.



* Method 1: Use definition

$\gamma(t) = 1 + 2e^{it}, \quad 0 \leq t \leq 2\pi$

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{1+2e^{it}}\, 2i\,e^{it}\,dt$$

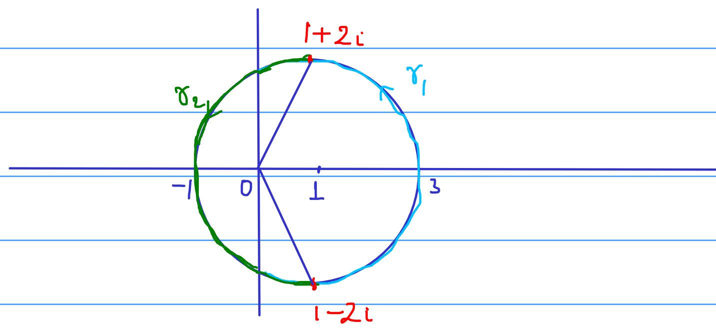
Substitution $u = 1 + 2e^{it}$ is not good because it will give back

$$\int_\gamma \frac{1}{u}\,du.$$

Trying to write integrand in standard form

$$\int_0^{2\pi} \frac{2i(\cos t + i\sin t)}{1+2\cos t + i\,2\sin t}\,dt = \cdots \quad \text{too complicated!}$$

* <u>Method 2</u>: try to find an antiderivative and use Fund. Thm. of Calc.
One needs to introduce a branch cut in order to find an antider. of $1/z$. However, every branch cut intersects γ.
Remedy: break γ into 2 parts. Then use different branch cut on each part.



$\gamma = \gamma_1 + \gamma_2$

$$\int_{\gamma_1} \frac{1}{z}\,dz = \operatorname{Log} z \Big|_{1+2i}^{1-2i} = \operatorname{Log}(1-2i) - \operatorname{Log}(1+2i) = \cdots$$

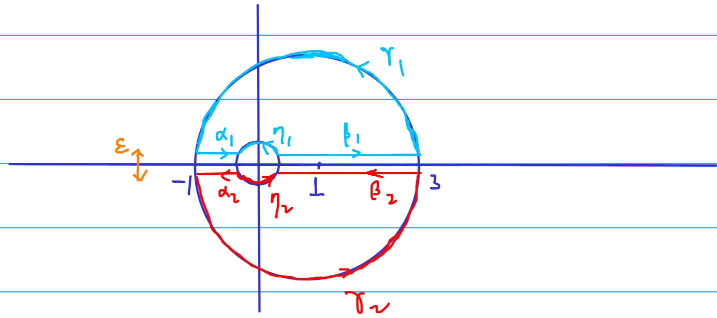
$$\int_{\gamma_2} \frac{1}{z}\,dz = \mathcal{L}og\, z \Big|_{1+2i}^{1-2i} = \mathcal{L}og(1-2i) - \mathcal{L}og(1+2i) = \cdots$$

where $\mathcal{L}og\, z$ is the branch of logarithm in which the argument is chosen to be in $(0, 2\pi)$.

$$\arg(1-2i) = \arctan\left(\frac{-2}{1}\right) + 2\pi = \cdots$$

Then $$\int_\gamma \frac{1}{z}\,dz = \int_{\gamma_1} \cdots + \int_{\gamma_2} \cdots$$

* <u>Method 3</u> : (idea of Cauchy)



Cut γ into γ_1, γ_2 and two small gaps of size ε as in the picture. The small circle is centered at 0, with some radius r.

$\eta(t) = re^{it}, \ 0 \le t \le 2\pi$

$$\int_{\gamma_1} + \int_{\alpha_1} - \int_{\eta_1} + \int_{\beta_1} = \int_{\gamma_1 + \alpha_1 - \eta_1 + \beta_1} = 0$$

$$+ \quad \int_{\gamma_2} + \int_{\alpha_2} - \int_{\eta_2} + \int_{\beta_2} = \int_{\gamma_2 + \alpha_2 - \eta_2 + \beta_2} = 0$$

$$\underbrace{\left(\int_{\gamma_1} + \int_{\gamma_2}\right)}_{\to \int_\gamma} + \underbrace{\left(\int_{\alpha_1} + \int_{\alpha_2}\right)}_{\to 0 \text{ as } \varepsilon \to 0} - \underbrace{\left(\int_{\eta_1} + \int_{\eta_2}\right)}_{\to \int_\eta} + \underbrace{\left(\int_{\beta_1} + \int_{\beta_2}\right)}_{\to 0 \text{ as } \varepsilon \to 0} = 0$$

Thus,

$$\int_\gamma \frac{1}{z}\,dz = \underbrace{\int_\eta \frac{1}{z}\,dz}$$

$$= \int_0^{2\pi} \frac{1}{re^{it}}\, rie^{it}\,dt = 2\pi i$$

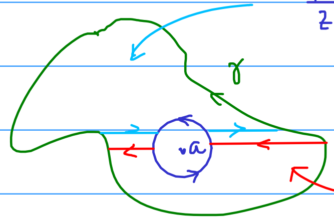
Cauchy's idea can be formulated in more generality as follows:

<u>Thm</u> (Cauchy's Integral Formula)

Let γ be simple loop positively oriented, G be the region enclosed by γ, point $a \in G$. Let $f: G \to \mathbb{C}$ be holomorphic. Then

$$\int_\gamma \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

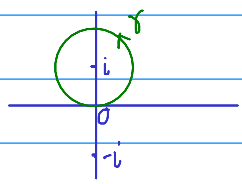
$\dfrac{f(z)}{z-a}$ is holomorphic



$\dfrac{f(z)}{z-a}$ is holomorphic

Ex:

$$\int_\gamma \frac{1}{z^2+1}\,dz = \int_\gamma \frac{1}{(z-i)(z+i)}\,dz = \int_\gamma \frac{\frac{1}{z+i}}{z-i}\,dz$$



Put $f(z) = \dfrac{1}{z+i}$ ... holomorphic in $D_1(i)$.

By Cauchy's Integral Formula,

$$\int_\gamma \cdots = 2\pi i\, f(i) = 2\pi i \,\frac{1}{2i} = \pi$$