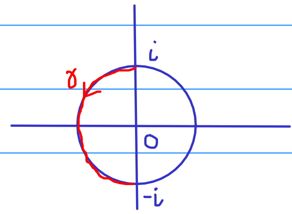
Lecture 23 (5/24/2019)

Ex:



$$\int_\gamma z^i \, dz = ?$$

* Wrong solution:

An antiderivative of z^i is $\dfrac{z^{i+1}}{i+1} = \dfrac{e^{(i+1)\mathcal{L}og\, z}}{i+1}$

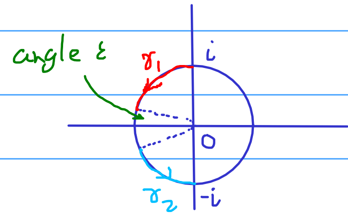
where $\mathcal{L}og\, z$ is the branch of logarithm corresponding to $\arg z \in [0, 2\pi)$.

Then $$\int_\gamma z^i \, dz = \left.\frac{z^{i+1}}{i+1}\right|_i^{-i} = \left.\frac{e^{(i+1)\mathcal{L}og\, z}}{i+1}\right|_i^{-i} = \cdots$$

The wrong step is in red. The function z^i given at the beginning is understood as $e^{i\,\mathrm{Log}\, z}$. That is, the principal logarithm is used. An antiderivative of z^i is $\dfrac{z^{i+1}}{i+1} = \dfrac{e^{(i+1)\,\mathrm{Log}\, z}}{i+1}$

In other words, one can't pick any other branch for the antiderivative. Therefore, one simply can't apply Fund. Thm. of Calc. to compute $\int_\gamma z^i dz$.

* Correct solution:



Cut γ to avoid the branch cut. For small $\varepsilon > 0$, let γ_1 be the arc from i to $e^{i(\pi-\varepsilon)}$, and γ_2 be the arc from $e^{i(\pi+\varepsilon)}$ to i.

Then $$\int_\gamma z^i dz = \lim_{\varepsilon \to 0} \left(\int_{\gamma_1} + \int_{\gamma_2} \right)$$

On each of γ_1 and γ_2, one can use the Fundamental Thm. of Calc.

$$\int_{\gamma_1} z^i\,dz = \frac{z^{i+1}}{i+1}\Bigg|_{i}^{e^{i(\pi-\varepsilon)}} = \frac{\left[e^{i(\pi-\varepsilon)}\right]^{i+1} - i^{i+1}}{i+1}$$

$$= \frac{e^{(i+1)\operatorname{Log} e^{i(\pi-\varepsilon)}} - e^{(i+1)\operatorname{Log} i}}{i+1}$$

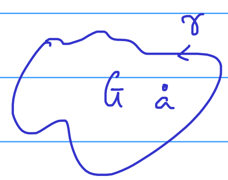
$$= \cdots$$

$$\xrightarrow{\varepsilon\to 0} \frac{-e^{-\pi} - i e^{-\pi/2}}{i+1}$$

Similarly, $\displaystyle\int_{\gamma_2} z^i\,dz \xrightarrow{\varepsilon\to 0} \frac{e^{\pi} - i e^{\pi/2}}{i+1}$

Thus, $\displaystyle\int_{\gamma} z^i\,dz = \frac{-e^{-\pi} - i e^{-\pi/2}}{i+1} + \frac{e^{\pi} - i e^{\pi/2}}{i+1} = \cdots$ (simplify if desire)

Recall Cauchy's Integral Formula:

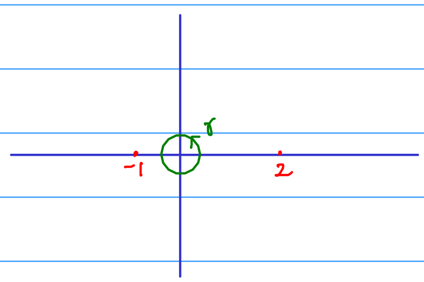


γ simple loop, positively oriented

f is holo. on G (region enclosed by γ) and cont. on $\bar{G}$ (closure of G).

$$\int_{\gamma} \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Ex: $\displaystyle\int_{C_r(0)} \frac{1}{(z+1)(z-2)}\,dz = ?$



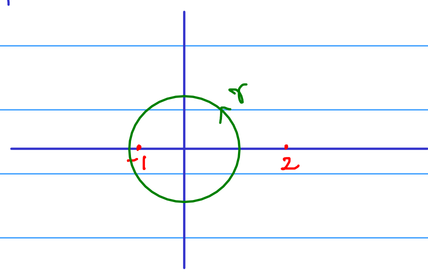
The integrand is holomorphic everywhere except at -1 and 2.

$\gamma = C_r(0)$ positively oriented.

- If $r<1$, the integrand is holomorphic in $D_r(0)$.

Thus, $\displaystyle\int_{\gamma} \cdots\,dz = 0$ by Cauchy-Goursat thm.

- If $r = 1$ then γ passes through -1, which is where the integrand is singular. We skip this case.
- If $1 < r < 2$ then the only singular point enclosed by γ is -1.

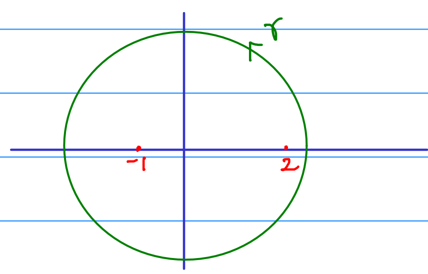


$$\int_\gamma \cdots dz = \int_\gamma \frac{\overbrace{\frac{1}{z-2}}^{f(z)}}{z+1}\, dz$$

Note that $f(z)$ is holo. in $D_r(0)$. By Cauchy's Integral Formula,

$$\int_\gamma \cdots dz = \int_\gamma \frac{f(z)}{z+1}\, dz = 2\pi i\, f(-1) = 2\pi i\, \frac{1}{-1-2} = -\frac{2\pi}{3} i$$

- If $r = 2$: singularity issue.
- If $r > 2$:



There are two ways to deal with this situation:

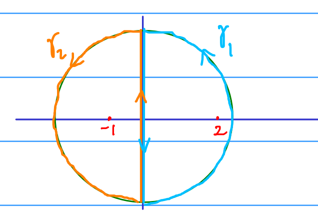
① Use partial fraction : this method can be used for any case of r, not just for $r > 2$.

$$\frac{1}{(z-2)(z+1)} = \frac{1}{3}\left(\frac{1}{z-2} - \frac{1}{z+1}\right)$$

$$\int_\gamma \cdots = \frac{1}{3} \underbrace{\int_\gamma \frac{1}{z-2}\, dz}_{2\pi i} - \frac{1}{3} \underbrace{\int_\gamma \frac{1}{z+1}\, dz}_{2\pi i} = 0$$

② Isolate the singular points:

We separate the singular points $-1, 2$ by a cut as in the picture.



$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2}$$

Note that each of γ_1 and γ_2 encloses only one singular point. Then we apply Cauchy's Integral Formula to each curve:

$$\int_{\gamma_1} \frac{1}{(z-2)(z+1)}\,dz = \int_{\gamma_1} \frac{\overbrace{\frac{1}{z+1}}^{f(z)}}{z-2}\,dz = 2\pi i f(2) = \frac{2\pi}{3} i$$

$$\int_{\gamma_2} \frac{1}{(z-2)(z+1)}\,dz = \int_{\gamma_2} \frac{\overbrace{\frac{1}{z-2}}^{g(z)}}{z+1}\,dz = 2\pi i g(-1) = -\frac{2\pi}{3} i$$

Thus, $\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} = 0$.

* Note: this method seems ad hoc, but we will see that it is a systematic approach known as Calculus of Residue.

<u>Some consequences of Cauchy's Integral Formula</u>

Cauchy's formula triggers a series of surprising results in the Calculus of Complex variables, which is not found in the Real variables. This formula is therefore a unique feature of Complex variables.

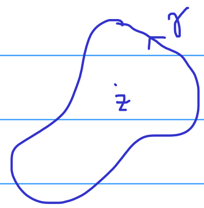
① Suppose that f is <u>holomorphic</u> on a region G enclosed by a simple loop γ, and that f is <u>continuous</u> on $\overline{G} = G \cup \gamma$.

By Cauchy's formula,

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz$$

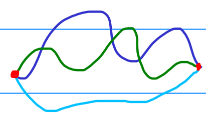
Let's change name: "z" to "w" and "a" to "z".

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\,dw$$



In order to compute the right hand side, one only needs to know the value of f on γ. However, z on the left hand side can be any point inside γ.
Thus, a holomorphic function on a region is completely determined by its value on the boundary of that region.

This property is certainly not true for Real variables: one can perturb a smooth function, while fixing its values on the boundary, and still get a smooth function.



(2) Differentiate both sides of the formula:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw$$

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^2}\, dw$$

There is no need to worry about the denominator being 0 because z lies strictly inside γ.

$$f''(z) = \frac{2}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^3}\, dw$$

....

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw \qquad (*)$$

We see that a holomorphic function is infinitely differentiable. Every derivative of its is holomorphic.
This phenomenon is of course not true in Real Variables. For example, function $f(x) = |x|x$ has f' but not f''.

$$C^1(\mathbb{R}) \supset C^2(\mathbb{R}) \supset C^3(\mathbb{R}) \supset \cdots \supset \underbrace{C^k(\mathbb{R})} \supset \cdots\cdots$$

k times differentiable functions, with continuous k'th derivatives.

For Complex variables, the whole chain collapses to one:

$$C^1(\mathbb{C}) = C^2(\mathbb{C}) = C^3(\mathbb{C}) = \cdots$$

Another way to write $(*)$ is the following:

$$\boxed{\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)} \qquad (**)$$

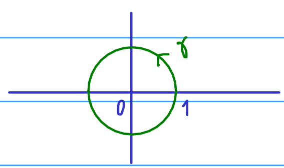
③ If $f: G \to \mathbb{C}$ has an antiderivative then f itself must be holomorphic.

Why? $f = F'$

F is holomorphic. Thus, every derivative of F is also holomorphic.

Ex: $f(z) = x = \mathrm{Re}(z)$ has no antider. because it's not holomorphic (by Cauchy-Riemann eqs.)

Ex:



$$\int_\gamma \frac{\sin z}{z^2}\,dz = ?$$

Apply $(**)$ for $f(z) = \sin z$, $n = 1$, $a = 0$:

$$\int_\gamma \frac{\sin z}{z^2}\,dz = 2\pi i\, f'(0) = 2\pi i \cos 0 = 2\pi i$$