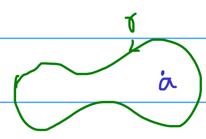
Lecture 24 (5/29/2019)

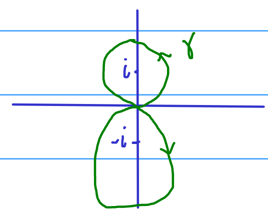
Recall the general Cauchy's Integral Formula:

$$\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

where γ is a simple loop positively oriented, a is encircled by γ, and f is holomorphic on the region enclosed by γ.

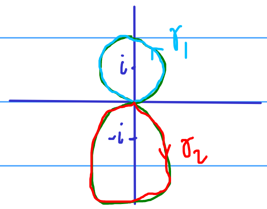


Ex:



$$\int_\gamma \frac{1}{(z^2+1)^2}\,dz = ?$$

We see that γ is not a simple curve. We decompose the integral as follows:



$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2}$$

(although γ is not necessarily equal to $\gamma_1 + \gamma_2$)

γ_1 and γ_2 are simple loops.

γ_1 is positively oriented, γ_2 is negatively oriented.

Factor the integrand:

$$\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}$$

which is holomorphic on $\mathbb{C} \setminus \{\pm i\}$.

γ_1 encircles only one singular point, namely i.

$$\int_{\gamma_1} \frac{1}{(z+i)^2(z-i)^2}\,dz = \int_{\gamma_1} \frac{f(z)}{(z-i)^2}\,dz$$ (all singularity is pushed to the denominator.)

where $f(z) = \frac{1}{(z+i)^2}$, which is holomorphic inside γ_1.

With $n=1$, $f'(z) = -2(z+i)^{-3}$

$$\int_{\gamma_1} \frac{f(z)}{(z-i)^2}\,dz = \frac{2\pi i}{1!} f'(i) = -4\pi i\,(2i)^{-3} = \frac{\pi}{2}$$

Similarly,

$$\int_{\gamma_2} \frac{1}{(z-i)^2(z+i)^2}\,dz = \int_{\gamma_2} \frac{g(z)}{(z+i)^2}\,dz$$ (where $g(z) = \frac{1}{(z-i)^2}$)

$$= -\frac{2\pi i}{1!} g'(i)$$ (minus sign because γ_2 is negatively oriented)

$$= -\frac{\pi}{2}$$

Thus, $\int_\gamma = \frac{\pi}{2} + \left(-\frac{\pi}{2}\right) = 0.$

* Reflect on the methods we have learned to compute

$$\int_\gamma f(z)\,dz$$

① Numerical approximation (analogous to left/right/middle Riemann sum)

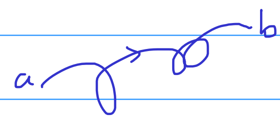
$$\approx \sum_{k=0}^{n} f(z_k^*)\,(z_{k+1} - z_k)$$

② Parametrizing the path:

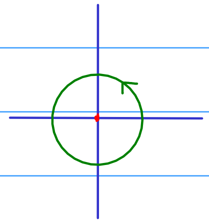
$$\int_\alpha^\beta f(\gamma(t))\,\gamma'(t)\,dt$$

③ Find antiderivative F:

$$F(b) - F(a)$$



Ex:



$$\int_{C_1(0)} \frac{1}{z^2}\,dz = \left(-\frac{1}{z}\right)\Big|_1^1 = 0$$

Although γ encircles the origin, where the integral is singular, the integral over γ is still equal to 0 because $1/z^2$ has an antiderivative on $\mathbb{C}\setminus\{0\}$ and $\gamma \subset \mathbb{C}\setminus\{0\}$.

This result can also be explained by Cauchy's formula with $f(z) \equiv 1$, $n = 1$.

④ Cauchy's formula:

$$\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \pm \frac{2\pi i}{n!} f^{(n)}(a)$$

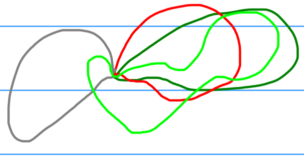
The plus sign is chosen if γ is positively oriented. The minus sign is chosen if γ is negatively oriented.

The first three methods work for any path, closed or not. The fourth method only works for simple loops. Why is integral over simple loops interesting? The only simple loop on the real line is a point. In other words, the concept of "loops" is trivial on the real line. Integral over any loop is zero:

$$\int_a^a f(x)\,dx = 0 \qquad (*)$$

This is a trivial identity, even regarded as a convention.

However, "loop" is a meaningful concept on the complex plane. For example, there are infinitely many simple loops passing by the same point.

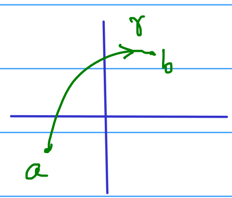


Therefore, an equality $\int_\gamma f(z)\,dz = 0$ is desirable, as an analogy to $(*)$

in real variable case. We see from Cauchy's Integral formula that integral over a simple loop of a complex function is not always zero. This feature is unique to Complex Variables, which implies many other surprising results. For example, if a function is holomorphic, it is infinitely differentiable. Another example is that if a function has an antiderivative then the function itself must be holomorphic.

* We will investigate the full strength of Method 4. Consider the following examples:

<u>Ex</u>:



$$\int_\gamma \frac{1}{z}\,dz = \mathcal{L}og\, z \Big|_a^b = \mathcal{L}og\, b - \mathcal{L}og\, a$$

where $\mathcal{L}og$ the branch of Logarithm with $\arg \in [0, 2\pi)$.

$$\int_\gamma \frac{1}{z^2}\,dz = -\frac{1}{z}\Big|_a^b = -\frac{1}{b} - \left(-\frac{1}{a}\right)$$

In general, for all $n \neq -1$,

$$\int_\gamma z^n\,dz = \frac{1}{n+1} z^{n+1}\Big|_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}$$

Thus, if $f(z)$ is decomposed into power functions z^n $(n \in \mathbb{Z})$ then we know how to compute $\int_\gamma f(z)\,dz$.

This motivates the study of power representation of complex functions:

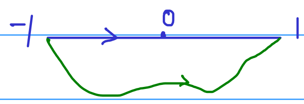
$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$ Taylor series about 0, or Maclaurin series

$f(z) = \dots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + \dots$ Laurent series about 0.

Why do we consider negative powers?

Negative powers are not of interest as one considers real integrals due to singularity issue.

$$\int_{-1}^{1} \frac{1}{x^2}\, dx \quad \text{is not defined}$$



There is only one path from -1 to 1, which passes by 0, where the integrand is singular.

However, there are many ways on the complex plane to go from -1 to 1 that avoid 0. Thus, the integral $\int_\gamma \frac{1}{z^2}\, dz$ is meaningful, as long as γ doesn't pass through 0.

<u>Def</u>: the series $\sum_{n=0}^{\infty} a_n$ is defined as $\lim_{n\to\infty} \underbrace{\sum_{k=0}^{n} a_k}_{\text{partial sum}}$.

Note that two ingredients needed in this definition are: addition and limit. Both of them exist in complex numbers. Thus, the most part of literature on real series is rewritten verbatim for complex series.

<u>Def</u>: the series $\sum a_n$ is said to converge absolutely if $\sum |a_n| < \infty$.

* Convergence tests:

- Divergence test: if $\sum a_n$ converges then $\lim a_n = 0$.
- Comparison test: if $|a_n| \le b_n$ and $\sum b_n < \infty$ then $\sum a_n$ converges absolutely.

- Ratio test:

  Suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$.

  If $L > 1$: $\sum a_n$ diverges

  If $L < 1$: $\sum a_n$ converges

  If $L = 1$: test fails

- Root test:

  Suppose $\lim \sqrt[n]{|a_n|} = L$.

  The same conclusions as in Ratio test.

* Power series is a series of the form $\sum_{n=0}^{\infty} a_n z^n$.

By Ratio test or Root test, the radius of convergence of this series is

$$R = \frac{1}{L}$$

where $L = \lim \frac{|a_{n+1}|}{|a_n|}$ (if using Ratio test)

$= \lim \sqrt[n]{|a_n|}$ (if using Root test)

Ex: $$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

$$\frac{z}{4-z} = \frac{z}{4}\,\frac{1}{1-\frac{z}{4}} = \frac{z}{4}\left(1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \left(\frac{z}{4}\right)^3 + \cdots\right)$$