Lecture 25 (5/31/2019)

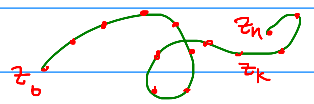
Return to the question how to compute $\int_\gamma f(z)\,dz$.

If f is complicated, finding an antiderivative is difficult. If the parametrization of γ is complicated, the formula

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\,\gamma'(t)\,dt$$

is not helpful either. One needs a numerical method to evaluate the integral. We have seen such a method: Riemann sum.

$$\int_\gamma f(z)\,dz \approx \sum_{k=0}^{n-1} f(z_k)(z_{k+1}-z_k)$$



Note that in this method we don't care about the integrand, only the curve. On the other hand, we know that in many cases, the integral doesn't depend on the path itself, only on the endpoints. By using Riemann sum method, one may have computed numerous quantities that do not matter: the $f(z_k)$ where z_k lies between z_0 and z_n.

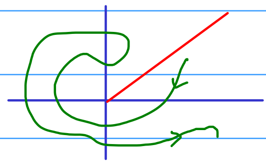
Another approach is to focus on the integrand rather than the curve. The idea is to decompose the integrand into simpler "modes":

$$f(z) = \sum a_k \underbrace{z^k}_{\text{simple mode}}$$

The integral $\int_\gamma z^k\,dz$ is quite simple to compute.

If $k \neq -1$ then $\displaystyle\int_\gamma z^k\,dz = \frac{z^{k+1}}{k+1}\Big|_a^b = \frac{b^{k+1}-a^{k+1}}{k+1}$

If $k=-1$ then one needs to be careful, for example choose a suitable branch for the logarithm.



This "mode decomposition" method doesn't care too much about the curve (with only one exeption for mode z^{-1}). The only information that matter is how to represent function $f(z)$ as a power series. We will continue to investigate power series with this motivation in mind.

Each power series $\sum_{k=0}^{\infty} a_k z^k$ has a radius of convergence R such that:

- if $|z| < R$, the series converges absolutely.
- if $|z| > R$, the series diverges.

How to find R? Use Ratio test or Root test.

Ex: $$\sum_{k=0}^{\infty} \underbrace{\frac{z^{2k}}{2^k k}}_{b_k}$$

Consider quotient $$\frac{b_{k+1}}{b_k} = \frac{z^{2(k+1)}}{2^{k+1}(k+1)} \frac{2^k k}{z^{2k}} = \frac{z^2}{2} \frac{k}{k+1}$$

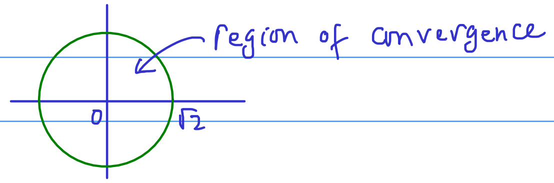
$$\longrightarrow \frac{z^2}{2} \text{ as } k \to \infty.$$

Then $$\lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right| = \frac{|z|^2}{2} = L$$

If $L < 1$: abs. convergence

If $L > 1$: divergence

Thus, the radius of convergence is $R = \sqrt{2}$.
The region of convergence is defined as $D_R(0)$, which is the disk $D_{\sqrt{2}}(0)$ in this case. This is analogous to "interval of convergence" $(-R, R)$ in real series.



The literature of complex series is very similar to that of real series because they are mostly founded on two structures: algebraic structure (addition, multiplication, ...) and distance (limit). The proofs are carried out the same way, almost verbatim. However, any results in real series that involves ordering of numbers doesn't have its complex counterpart because there is no ordering on the complex plane. For example, the Alternating Series test

$$\sum (-1)^k a_k$$

requires a_k to be nonnegative and decreasing.

* Derivative of power series:
  With the region of convergence, the power series defines a holomorphic function:

$$f(z) := \sum_{k=0}^{\infty} a_k z^k$$

$$\Rightarrow f'(z) = \sum_{k=1}^{\infty} a_k k z^{k-1} \qquad \forall z \in D_R(0)$$

In other words, in $D_R(0)$ one can differentiate the series term by term. The radius of convergence of the derivative series is also equal to R. Why?

$$\left|\frac{a_{k+1}(k+1)}{a_k k}\right| = \left|\frac{a_{k+1}}{a_k}\right| \frac{k+1}{k} \longrightarrow \frac{1}{R} \quad \text{as } k\to\infty$$

We know that it is very hard for a complex-variabled function to be differentiable (since it has to satisfy Cauchy-Riemann eqs.) It seems too easy for the function $f(z)$ above to be differentiable, since one doesn't need to check C-R eqs. The key is that function $f(z)$ was given as a power series. This form gives it an advantage.

One can continue to differentiate within $D_R(0)$:

$$f''(z) = \sum_{k=2}^{\infty} a_k k(k-1) z^{k-2} \qquad \forall z \in D_R(0).$$

$\cdots\cdots$

Ex: Express $\dfrac{1}{(z-2)^2}$ as a power series.

$$\frac{1}{(z-2)^2} = \left(-\frac{1}{z-2}\right)' = \left(\frac{1}{2-z}\right)'$$

We have

$$\frac{1}{2-z} = \frac{1}{2}\,\frac{1}{1-z/2} = \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right)$$

$$= \frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\frac{z^3}{2^4}+\cdots$$

$$= \sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}}$$

This series representation is valid only if $\left|\frac{z}{2}\right|<1$, i.e. $|z|<2$.
Now take the derivative both sides:

$$\underbrace{\left(\frac{1}{2-z}\right)'}_{\frac{1}{(z-2)^2}} = \sum_{k=1}^{\infty}\frac{k z^{k-1}}{2^{k+1}} \qquad \forall z \in D_2(0)$$

We have seen that a power series defines a holomorphic function. A natural question is: can any holomorphic function be expressed as a power series? The answer is yes. This is a surprising result since it's not the case for real-variabled functions.

Suppose a holomorphic function $f(z)$ can be written as a power series

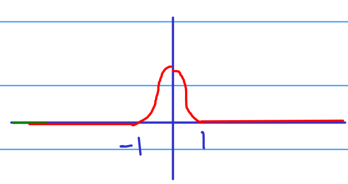
$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

One can check that

$$f(0) = a_0 \quad \Rightarrow \quad a_0 = f(0)$$
$$f'(0) = a_1 \quad \Rightarrow \quad a_1 = f'(0)$$
$$f''(0) = 2a_2 \quad \Rightarrow \quad a_2 = \frac{1}{2} f''(0)$$
$$\dots$$
$$f^{(k)}(0) = k!\, a_k \quad \Rightarrow \quad a_k = \frac{1}{k!} f^{(k)}(0)$$

Therefore, if a function can be written as a power series, this series is unique, and is called Taylor series of f (about 0).

Now consider a real function $f(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{if } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$



This is a "bump" function. It is infinitely differentiable anywhere in $\mathbb{R}$. However, it doesn't have Taylor series representation at -1. Indeed, suppose that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(-1)}{k!} (x+1)^k$$

Since f is a constant function on the left of -1, derivatives $f^{(k)}(-1) = 0$ for every k. The RHS is therefore equal to 0 for any x near -1.

This is a contradiction because the LHS is $f(x)$, which is nonzero when x is on the right of -1.

We will see that this phenomenon doesn't happen for complex-variabled functions. A holomorphic function can always be expressed as a power series.